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Ward Identities and Vanishing of the Beta Function for $d=1$ Interacting Fermi Systems

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We give a self consistent and simplified proof of the (asymptotic) vanishing of the Beta function in $d=1$ interacting Fermi systems as a consequence of a few properties deduced from the exact solution of the Luttinger model. Moreover, since the vanishing of the Beta function is usually "proved" in the physical literature through heuristic arguments based on Ward identities, we briefly discuss here also the possibility of exploiting this idea in a rigorous approach, by using a suitable Dyson equation. We show that there are serious difficulties, related to the presence of corrections (for which we get careful bounds), which are usually neglected.

KEY WORDS: Fermion systems; Ward Identities; renormalization group.

1. INTRODUCTION

Inspired by a previous deep analysis by Tomonaga, forty years ago Luttinger[13] introduced his model, describing two kinds of $d=1$ fermions with linear dispersion relation and interacting via a short range potential, as a model for $d=1$ metals. The solution of the model given in ref. 13 was however incorrect and a true solution was given a bit later by Mattis and Lieb.[20] They showed that the model can be exactly reexpressed in terms of a free bosonic model; this implies that all the correlation functions of the model can be computed (an explicit expression can be found in ref. 3). However the solution given in ref. 20 seems to depend crucially from the details of the Luttinger model Hamiltonian, and even apparently harmless and physically irrelevant modifications of it completely destroy the possibility of an exact mapping into a free bosonic model. This is unlucky,

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because a large number of important models, for which an exact solution is lacking (at least for the correlation functions), can be reexpressed in terms of interacting fermionic models and it is physically very reasonable to assume that they are in the same class of universality of the Luttinger model (or of its massive version, which is however *not* solvable). We mention the Heisemberg models for spin $\frac{1}{2}$ quantum spin chains, like the XXZ or the XYZ chain, see refs. 1 or 15, or classical bidimensional spin lattice models, like the *Eight vertex* or the *Ashkin–Teller* models, see refs. 15 or 21. More recent examples are models of vicinal surfaces[24] or the domain wall theory of commensurate-incommensurate phase transitions in two dimension.[22] In all such cases the mapping into a fermionic theory, and the subsequent assumption that it belongs to the same class of universality of the massless or massive Luttinger model, seems a very powerful method (sometimes the only one) to get information about the asymptotic behavior of the correlations.

In the last decade, starting from,[2] a perturbative approach based on Renormalization Group methods has indeed achieved the goal of using the Luttinger model exact solution to get the asymptotic behavior of the correlations for many of such models at low temperature (for a recent review, see ref. 11). Among recent achievements is the computation in ref. 5 and 6 of the spin-spin correlation along the third axis of the XYZ model (with a possible inclusion of next nearest neighbor interaction) in a magnetic field; and the computation in refs. 16 and 17 of the energy-energy correlation and the specific heat near the critical point of many classical spin models coupled by quartic interactions, including the Eight-vertex and the Ashkin-Teller model. Such results convert (at least partially) into rigorous proofs the deep physical intuitions in refs. 1, 15, or 21.

The analysis starts by writing the generating function of the correlations as a Grassmann integral, and by expressing the Grassmanian integration as the product of many independent integrations, each of them describing the theory at a certain momentum scale. This allows us to perform the overall functional integration by iteratively integrating the Grassmanian variables of decreasing momentum scale. After each integration step one gets an effective theory similar to the initial one, the main differences being that the remaining Grassmanian integration is *renormalized* (the renormalization being defined in terms of a few parameters, *renormalization constants*, related to the critical indices of the model) and the *relevant* part of the interaction (which is described in terms of a few other parameters, the *running coupling constants*) (RCC in the following) is modified. The method works if the RCC remain small at each step; in fact, in this case, the correlations and the critical indices can be written as *convergent* power series in the RCC.

The RCC obey a complicated set of recursive equations, whose right hand side will be called, as usual, the Beta function. In order to prove that they do not grow as the momentum scale goes to zero, the Beta function is decomposed into two terms; the first term is common to all such models and is essentially equal to the Luttinger model Beta function, while the other one is model dependent. It turns out that the first term is asymptotically vanishing, even if the dimensional bounds following from the multiscale analysis do not support this result; we call this property "vanishing of Luttinger model Beta function." On the contrary, the dimensional bounds are sufficient to control the effect of the second term on the flow of the RCC in all the models we can successfully analyze, so that, by using a suitable iterative procedure, it is possible to show that the RCC stay indeed small on all scales. The details on the bound of the second term and on its (of course essential) role in explaining the physical properties of the different models, are given in the papers referenced before. The point is that a property valid for the Luttinger model (the vanishing of the Beta function) is used to prove that the RCC remain small (and so the expansion for the correlation function is convergent) in a number of not solvable models.

The main result of this paper is the proof of the vanishing of the Luttinger model Beta function (see Theorem (3.1)), which is at the core of the above results. Although a proof of this crucial point is already sketched in the literature, see refs. 4 and 7, it is in some point unnecessarily complicated and not all the details are published. We present here a new proof, which is based on the same ideas but which is much simpler. We shall give all the details, except those which can be taken from ref. 5 without any further discussion, as the bound (2.43) and Lemma 3.2.

The main technical difficulty in proving this result is that the Beta function is written by a convergent expansion and each order is obtained by summing up a certain number of terms; the vanishing of the Beta function is a consequence of certain complicated cancellations occurring at every order. While one can easily check by direct computation that such cancellations occur at lowest orders, to prove that they occur at every order looks to us essentially impossible. Our proof is instead based on the analyticity properties of the correlation functions of the Luttinger model as functions of the interaction λ and of a parameter δ describing the difference between the Fermi velocity and an arbitrary fixed value, say 1; such properties are deduced by the Luttinger model exact solution in refs. 3 and 20.

The proof is in Sections 2 and 3. In particular in Section 2 we present our Renormalization Group analysis of the Luttinger model with a local interaction, a fixed ultraviolet cutoff and an arbitrary infrared cutoff. Note that the interaction locality for the model with ultraviolet cutoff is chosen only for convenience, as any short range interaction would produce similar results. The outcome of the RG analysis is that the Schwinger functions are represented as expansions in terms of two sequences of RCC, one related with the interaction strength at different momentum scale, the other with the Fermi velocity (there is no renormalization of the Fermi momentum in the Luttinger model); if the RCC are small the expansions are convergent, and by them very careful bounds on the large distance asymptotic behavior are obtained. A similar analysis can be repeated for the Luttinger model with no cut-offs, due to the results in refs. 5 and 9, as explained in Section 3.2

In Section 3 we prove that indeed the RCC are bounded. First of all, we remark that, if the RCC of the Luttinger model are small, the same is true for the RCC of the Luttinger model with cut-off. This is due to the fact that the Beta functions of the Luttinger model with or without cut-off differ by terms which go to zero exponentially, if the momentum scale go to zero. We consider then the RCC of the Luttinger model in a finite volume L and we show, as a consequence of the analysis in Section 2, that there is a quantity, depending only on two and four points Schwinger functions computed at distances of order L, which is proportional to the running coupling constant which measures the effective interaction strength on scales of order L^{-1}, up to small corrections, see Lemma 3.3. The crucial point is that such quantity can be also computed by the explicit expression of the Luttinger model Schwinger functions, obtained from the exact solution (which is valid also at a finite volume L), and it turns out that such quantity is of order λ, see (3.12). This, togheter with the fact that the RCC for the model in a volume L are close to the one in the infinite volume (see (3.33), implies that the RCC of the infinite volume Luttinger model are well defined and of order λ on all scales, see Lemma 3.4. In this proof an important role is played by a Ward identity, see (3.40), which allows us to control the Fermi velocity renormalization in terms of the interaction strength renormalization. Finally, there is a simple argument, explained at the end of Section 3, proving the vanishing of the Beta function for the Luttinger model (and so for the model with infrared and ultraviolet cut-off) as a consequences of the above results on the RCC; this completes our proof.

In Section 4 we discuss the interesting question if the vanishing of the Beta function can be proved without any use of the Luttinger model exact solution, but directly in the framework of a functional integral analysis. The interest of such problem is in the possibility that the methods used to get this result could be extended to other problems, where an exact solution is missing, like in $d > 1$. A positive answer to this question was given in the physical literature, by a clever combination of Ward identities and Dyson equations in a Renormalization Group scheme, see refs. 8, 23, and in particular.[18, 19] However the presence of cutoffs in all the models one is interested in (the lattice or a non linear dispersion relation) breaks necessarily the gauge invariance by adding corrections to the Ward identities, which are neglected in the physical literature (they correspond to next to leading corrections), but should be taken into account in a rigorous approach. We have recently set up a formalism, see refs. 5 and 6, which allows us to derive Ward identities rigorously and to obtain careful bounds on the corrections; we used them to prove the vanishing of the density-density critical index in the XYZ model, so proving also a conjecture in ref. 24.

Encouraged by this result, we try to mimic the heuristic proof of the vanishing of the Beta function, by taking into account the corrections due to the cutoffs. We are indeed able to write a Dyson equation which takes into account the effect of the ultraviolet cutoff, and we get careful (essentially optimal) bounds on all the terms appearing in our Ward identity; a brief description of these attempts is given in Section (3). It turns out that, if one *neglects* the corrections due to the ultraviolet cutoffs, our Dyson equation reduces to the one in ref. 19, used to deduce the vanishing of the Beta function from dimensional arguments, that our analysis make rigorous. Hence, if we could prove that such corrections are indeed *negligible* in a suitable sense, we would get the expected result for our question.

However, this is *not* the case; we perform a rather detailed analysis of the corrections and we conclude that they are *not* negligible, in the sense that their presence prevents the possibility of proving the vanishing of the Beta function. The only open possibility, as we think that our estimates for the corrections can not be improved, is that it would be possible to make the corrections negligible, by moving the ultraviolet cutoff to infinity and by renormalizing the model, so that the effective interaction on momentum scale 1 stays bounded. This conjecture is based on the remark that the Ward identities are *formally exact* in the limiting theory (here the interaction locality is essential) and on the fact that the conjecture has been proved, at level of perturbation theory, in a similar problem.[12] To prove this conjecture is not a simple task, as it is equivalent to study the ultraviolet problem in a relativistic quantum field theory (the *Thirring* model) and we did not yet face it seriously, but the analysis of Section 4 would certainly be an essential step. This is why we decided to publish here at least a sketch of our actual results.

We now give a brief outline of the following sections. In Section 2.1 we define precisely the Luttinger model with cutoff. In Section 2.2 we describe the corresponding RG expansion for the Schwinger functions and the definition of the Beta function; the main point in this section is equation (2.42) and the corresponding bound (2.43). The vanishing of Luttinger Beta function is proved in Section 3, using the arguments discussed before.

The gauge transformation and the corresponding Ward identities are discussed in Section 4.1, while the Dyson equation is described in Section 4.2. The consequences of Ward identities and Dyson equation and their relation with the main problem of this paper are briefly discussed in Section 4.3. Some bounds used in this discussion are proved in the Appendix A1.

2. RENORMALIZATION GROUP ANALYSIS

2.1. The Model

We consider a one dimensional system of two kinds of fermions with linear dispersion relation and interacting with a local potential. The presence of an ultraviolet and infrared cutoff makes the model *not* solvable; if the cutoffs are removed and the local potential is replaced by a short-ranged one the model coincides with the Luttinger model.

Given the interval $[0, L]$, the inverse temperature β and the (large) integer N, we introduce in $\Lambda = [0, L] \times [0, \beta]$ a lattice Λ_N, whose sites are given by the *space-time points* $\mathbf{x} = (x, x_0) = (na, n_0 a_0)$, $a = L/N$, $a_0 = \beta/N$, $n, n_0 = 0, 1,..., N-1$. We also consider the set $\mathscr{D}$ of *space-time momenta* $\mathbf{k} = (k, k_0)$, with $k = \frac{2\pi}{L}(n+\frac{1}{2})$ and $k_0 = \frac{2\pi}{\beta}(n_0+\frac{1}{2})$, $n, n_0 = 0, 1,..., N-1$. With each $\mathbf{k} \in \mathscr{D}$ we associate four Grassmanian variables $\hat{\psi}^{[h,0]\sigma}_{\mathbf{k},\omega}$, $\sigma, \omega \in \{+, -\}$, where h is a negative integer related with the infrared cutoff, see below. The lattice Λ_N is introduced only for technical reasons, so that the number of Grassmann variables is finite, and eventually the limit $N \to \infty$ is taken (and it is trivial, see refs. 5). Then we define the functional integration $\int \mathscr{D}\psi^{[h,0]}$ as the linear functional on the Grassmann algebra generated by the variables $\hat{\psi}^{[h,0]\sigma}_{\mathbf{k},\omega}$, such that, given a monomial $Q(\hat{\psi})$ in the variables $\hat{\psi}^{[h,0]\sigma}_{\mathbf{k},\omega}$, its value is 0, except in the case $Q(\hat{\psi}) = \prod_{\mathbf{k} \in \mathscr{D}, \omega = \pm} \hat{\psi}^{[h,0]-}_{\mathbf{k},\omega} \hat{\psi}^{[h,0]+}_{\mathbf{k},\omega}$, up to a permutation of the variables. In this case the value of the functional is determined, by using the anticommuting properties of the variables, by $\int \mathscr{D}\psi^{[h,0]} Q(\hat{\psi}) = 1$. We also define the *Grassmanian field* on the lattice Λ_N as

$$\psi^{[h,0]\sigma}_{\mathbf{x},\omega} = \frac{1}{L\beta} \sum_{\mathbf{k} \in \mathscr{D}} e^{i\sigma\mathbf{k}\mathbf{x}} \hat{\psi}^{[h,0]\sigma}_{\mathbf{k},\omega}, \qquad \mathbf{x} \in \Lambda_N. \tag{2.1}$$

Note that $\psi^{[h,0]\sigma}_{\mathbf{x},\omega}$ is antiperiodic both in time and space variables.

The *Schwinger functions* are defined by

$$S(\mathbf{x}_1, \sigma_1, \omega_1;...; \mathbf{x}_s, \sigma_s, \omega_s) = \frac{\int P(d\psi^{[h,0]})\, e^{-V(\psi^{[h,0]})} \prod_{i=1}^{s} \psi^{[h,0]\sigma_i}_{\mathbf{x}_i,\omega_i}}{\int P(d\psi^{[h,0]})\, e^{-V(\psi^{[h,0]})}}, \tag{2.2}$$

where

$$V(\psi^{[h,0]}) = \lambda \int d\mathbf{x}\, \psi^{[h,0]+}_{\mathbf{x},+} \psi^{[h,0]-}_{\mathbf{x},+} \psi^{[h,0]+}_{\mathbf{x},-} \psi^{[h,0]-}_{\mathbf{x},-} \tag{2.3}$$

and

$$P(d\psi^{[h,0]}) = \mathcal{N}^{-1} \mathcal{D}\psi^{[h,0]} \cdot \exp\left\{ -\frac{1}{L\beta} \sum_{\omega=\pm 1} \sum_{\mathbf{k}\in\mathcal{D}} C_{h,0}(\mathbf{k})(-ik_0+\omega k)\, \hat{\psi}^{[h,0]+}_{\mathbf{k},\omega} \hat{\psi}^{[h,0]-}_{\mathbf{k},\omega} \right\}, \tag{2.4}$$

with $\mathcal{N} = \prod_{\mathbf{k}\in\mathcal{D}} [(L\beta)^{-2}(-k_0^2-k^2)\, C_{h,0}(\mathbf{k})^2]$ and $\int d\mathbf{x}$ is a shorthand for "$aa_0 \sum_{\mathbf{x}\in\Lambda_N}$." The function $C_{h,0}(\mathbf{k})$ acts as an ultraviolet and infrared cutoff and it is defined in the following way. We introduce a positive number $\gamma > 1$ and a positive function $\chi_0(t) \in C^\infty(\mathbb{R}_+)$ such that

$$\chi_0(t) = \begin{cases} 1 & \text{if} \quad 0 \leqslant t \leqslant 1, \\ 0 & \text{if} \quad t \geqslant \gamma_0, \quad 1 < \gamma_0 \leqslant \gamma, \end{cases} \tag{2.5}$$

and we define, for any integer $j \leqslant 0$,

$$f_j(\mathbf{k}) = \chi_0(\gamma^{-j}\,|\mathbf{k}|) - \chi_0(\gamma^{-j+1}\,|\mathbf{k}|). \tag{2.6}$$

Finally we define

$$\chi_{h,0}(\mathbf{k}) = [C_{h,0}(\mathbf{k})]^{-1} = \sum_{j=h}^{0} f_j(\mathbf{k}), \tag{2.7}$$

so that $[C_{h,0}(\mathbf{k})]^{-1}$ is a smooth function with support in the interval $\{\gamma^{h-1} \leqslant |\mathbf{k}| \leqslant \gamma\}$, equal to 1 in the interval $\{\gamma^h \leqslant |\mathbf{k}| \leqslant 1\}$. In the following the ultraviolet cutoff is supposed fixed, while the infrared cutoff is supposed to vary and at the end we are interested in the limit $h \to -\infty$.

2.2. The Tree Expansion

We call $\psi^{[h,0]}$ simply ψ and we introduce the *generating functional*

$$\mathcal{W}(\phi, J) = \log \int P(d\psi)\, e^{-V(\psi) + \sum_\omega \int d\mathbf{x} [J_{\mathbf{x},\omega} \psi^+_{\mathbf{x},\omega} \psi^-_{\mathbf{x},\omega} + \phi^+_{\mathbf{x},\omega} \psi^-_{\mathbf{x},\omega} + \psi^+_{\mathbf{x},\omega} \phi^-_{\mathbf{x},\omega}]}. \tag{2.8}$$

The variables $\phi^\sigma_{\mathbf{x},\omega}$ are antiperiodic in x_0 and x and anticommuting with themselves and $\psi^\sigma_{\mathbf{x},\omega}$, while the variables $J_{\mathbf{x},\omega}$ are periodic and commuting with themselves and all the other variables. The Schwinger functions can be obtained by functional derivatives of (2.8); for instance

$$G^{2,1}_{\omega}(\mathbf{x};\mathbf{y},\mathbf{z})=\frac{\partial}{\partial J_{\mathbf{x},\omega}}\frac{\partial^2}{\partial\phi^+_{\mathbf{y},\omega}\,\partial\phi^-_{\mathbf{z},\omega}}\mathcal{W}(\phi,J)|_{\phi=J=0},\tag{2.9}$$

$$G^{2}_{\omega}(\mathbf{y},\mathbf{z})=\frac{\partial^2}{\partial\phi^+_{\mathbf{y},\omega}\,\partial\phi^-_{\mathbf{z},\omega}}\mathcal{W}(\phi,J)|_{\phi=J=0},\tag{2.10}$$

$$G^{4}_{\omega}(\mathbf{x}_1,\mathbf{x}_2,\mathbf{x}_3,\mathbf{x}_4)=\frac{\partial^2}{\partial\phi^+_{\mathbf{x}_1,\omega}\,\partial\phi^-_{\mathbf{x}_2,\omega}}\frac{\partial^2}{\partial\phi^+_{\mathbf{x}_3,-\omega}\,\partial\phi^-_{\mathbf{x}_4,-\omega}}\mathcal{W}(\phi,J)|_{\phi=J=0}.\tag{2.11}$$

The functional integration of the generating functional (2.8) can be performed iteratively in the following way. We prove by induction that, for any negative j, there are a constant E_j, a positive function $\tilde Z_j(\mathbf{k})$ and functionals $\mathcal{V}^{(j)}$ and $\mathcal{B}^{(j)}$ such that

$$e^{\mathcal{W}(\phi,J)}=e^{-L\beta E_j}\int P_{\tilde Z_j,C_{h,j}}(d\psi^{[h,j]})\,e^{-\mathcal{V}^{(j)}(\sqrt{Z_j}\,\psi^{[h,j]})+\mathcal{B}^{(j)}(\sqrt{Z_j}\,\psi^{[h,j]},\phi,J)},\tag{2.12}$$

where:

(1) $P_{\tilde Z_j,C_{h,j}}(d\psi^{[h,j]})$ is the *effective Grassmanian measure at scale j*, equal to, if $Z_j=\max_{\mathbf{k}}\tilde Z_j(\mathbf{k})$,

$$P_{\tilde Z_j,C_{h,j}}(d\psi^{[h,j]})=\prod_{\mathbf{k}:\,C_{h,j}(\mathbf{k})>0}\ \prod_{\omega=\pm1}\frac{d\hat\psi^{[h,j])+}_{\mathbf{k},\omega}\,d\hat\psi^{[h,j]-}_{\mathbf{k},\omega}}{\mathcal{N}_j(\mathbf{k})}$$
$$\cdot\exp\left\{-\frac{1}{L\beta}\sum_{\mathbf{k}}C_{h,j}(\mathbf{k})\,\tilde Z_j(\mathbf{k})\sum_{\omega\pm1}\hat\psi^{[h,j]+}_{\omega}\,D_\omega(\mathbf{k})\,\hat\psi^{[h,j]-}_{\mathbf{k},\omega}\right\},\tag{2.13}$$

$$\mathcal{N}_j(\mathbf{k})=(L\beta)^{-1}\,C_{h,j}(\mathbf{k})\,\tilde Z_j(\mathbf{k})[-k_0^2-k^2]^{1/2},\tag{2.14}$$

$$C_{h,j}(\mathbf{k})^{-1}=\sum_{r=h}^{j}f_r(\mathbf{k})\,\bar\chi_{h,j}(\mathbf{k}),\qquad D_\omega(\mathbf{k})=-ik_0+\omega k;\tag{2.15}$$

(2) the *effective potential on scale j*, $\mathcal{V}^{(j)}(\psi)$, is a sum of monomial of Grassman variables multiplied by suitable kernels, i.e., it is of the form

$$\mathcal{V}^{(j)}(\psi)=\sum_{n=1}^{\infty}\frac{1}{(L\beta)^{2n}}\sum_{\substack{\mathbf{k}_1,\dots,\mathbf{k}_{2n}\\ \omega_1,\dots,\omega_{2n}}}\prod_{i=1}^{2n}\hat\psi^{\sigma_i}_{\mathbf{k}_i,\omega_i}\hat W^{(j)}_{2n,\underline{\omega}}(\mathbf{k}_1,\dots,\mathbf{k}_{2n-1})\,\delta\left(\sum_{i=1}^{2n}\sigma_i\mathbf{k}_i\right),\tag{2.16}$$

where $\sigma_i=+$ for $i=1,\dots,n$, $\sigma_i=-$ for $i=n+1,\dots,2n$ and $\underline{\omega}=(\omega_1,\dots,\omega_{2n})$;

(3) the *effective source term at scale j*, $\mathcal{B}^{(j)}(\sqrt{Z_j}\,\psi, \phi, J)$, is a sum of monomials of grassman variables and $\phi^{\pm}$, J field, with at least one $\phi^{\pm}$ or one J field; we shall write it in the form

$$\mathcal{B}^{(j)}(\sqrt{Z_j}\,\psi, \phi, J) = \mathcal{B}_\phi^{(j)}(\sqrt{Z_j}\,\psi) + \mathcal{B}_J^{(j)}(\sqrt{Z_j}\,\psi) + W_R^{(j)}(\sqrt{Z_j}\,\psi, \phi, J), \tag{2.17}$$

where $\mathcal{B}_\phi^{(j)}(\psi)$ and $\mathcal{B}_J^{(j)}(\psi)$ denote the sums over the terms containing only one ϕ or J field, respectively.

Of course (2.12) is true for $j=0$, with

$$\tilde{Z}_0(\mathbf{k}) = 1, \qquad E_0 = 0, \qquad \mathcal{V}^{(0)}(\psi) = V(\psi), \qquad W_R^{(0)} = 0,$$

$$\mathcal{B}_\phi^{(0)}(\psi) = \sum_\omega \int d\mathbf{x} [\phi^+_{\mathbf{x},\omega}\psi^-_{\mathbf{x},\omega} + \psi^+_{\mathbf{x},\omega}\phi^-_{\mathbf{x},\omega}], \quad \mathcal{B}_J^{(0)}(\psi) = \sum_\omega \int d\mathbf{x}\, J_{\mathbf{x},\omega}\psi^+_{\mathbf{x},\omega}\psi^-_{\mathbf{x},\omega}. \tag{2.18}$$

Let us now assume that (2.12) is satisfied for a certain $j \leqslant 0$ and let us show that it holds also with $j-1$ in place of j.

In order to perform the integration corresponding to $\psi^{(j)}$, we write the effective potential and the effective source as sum of two terms, according to the following rules.

We split the effective potential $\mathcal{V}^{(j)}$ as $\mathcal{L}\mathcal{V}^{(j)} + \mathcal{R}\mathcal{V}^{(j)}$, where $\mathcal{R} = 1 - \mathcal{L}$ and $\mathcal{L}$, the *localization operator*, is a linear operator on functions of the form (2.16), defined in the following way by its action on the kernels $\hat{W}^{(j)}_{2n,\underline{\omega}}$.

(1) If $2n = 4$, then

$$\mathcal{L}\hat{W}^{(j)}_{4,\underline{\omega}}(\mathbf{k}_1, \mathbf{k}_2, \mathbf{k}_3) = \hat{W}^{(j)}_{4,\underline{\omega}}(\bar{\mathbf{k}}_{++}, \bar{\mathbf{k}}_{++}, \bar{\mathbf{k}}_{++}), \tag{2.19}$$

where $\bar{\mathbf{k}}_{\eta\eta'} = (\eta\pi L^{-1}, \eta'\pi\beta^{-1})$. Note that $\mathcal{L}\hat{W}^{(j)}_{4,\underline{\omega}}(\mathbf{k}_1, \mathbf{k}_2, \mathbf{k}_3) = 0$, if $\sum_{i=1}^4 \omega_i \neq 0$, by simple symmetry considerations.

(2) If $2n = 2$ (in this case there is a non zero contribution only if $\omega_1 = \omega_2$)

$$\mathcal{L}\hat{W}^{(j)}_{2,\underline{\omega}}(\mathbf{k}) = \frac{1}{4} \sum_{\eta,\eta'=\pm 1} \hat{W}^{(j)}_{2,\underline{\omega}}(\bar{\mathbf{k}}_{\eta\eta'}) \left\{ 1 + \eta \frac{L}{\pi} + \eta' \frac{\beta}{\pi} k_0 \right\}. \tag{2.20}$$

(3) In all the other cases

$$\mathcal{L}\hat{W}^{(j)}_{2n,\underline{\omega}}(\mathbf{k}_1, \ldots, \mathbf{k}_{2n-1}) = 0. \tag{2.21}$$

These definitions are such that $\mathscr{L}^2=\mathscr{L}$, a property which plays an important role in the analysis of ref. 5. Moreover, by using the symmetries of the model, it is easy to see that

$$\mathscr{L}\mathscr{V}^{(j)}(\psi^{[h,j]}) = z_j F_\zeta^{[h,j]} + a_j F_\alpha^{[h,j]} + l_j F_\lambda^{[h,j]}, \tag{2.22}$$

where z_j, a_j, and l_j are real numbers and

$$\begin{aligned} F_\alpha^{[h,j]} &= \sum_\omega \frac{\omega}{(L\beta)} \sum_{\mathbf{k}:\, C_{h,j}(\mathbf{k})>0} k \hat{\psi}^{[h,j]+}_{\mathbf{k},\omega} \hat{\psi}^{[h,j]-}_{\mathbf{k},\omega} \\ &= \sum_\omega i\omega \int_\Lambda d\mathbf{x}\, \psi^{[h,j]+}_{\mathbf{x},\omega} \partial_x \psi^{[h,j]-}_{\mathbf{x},\omega}, \end{aligned} \tag{2.23}$$

$$\begin{aligned} F_\zeta^{[h,j]} &= \sum_\omega \frac{1}{(L\beta)} \sum_{\mathbf{k}:\, C^{\varepsilon}_{h,j}(\mathbf{k})>0} (-ik_0)\, \hat{\psi}^{[h,j]+}_{\mathbf{k},\omega} \hat{\psi}^{[h,j]-}_{\mathbf{k}',\omega} \\ &= -\sum_\omega \int_\Lambda d\mathbf{x}\, \psi^{[h,j]+}_{\mathbf{x},\omega}\, \partial_0 \psi^{[h,j]-}_{\mathbf{x},\omega}, \end{aligned} \tag{2.24}$$

$$F_\lambda^{[h,j]} = \frac{1}{(L\beta)^4} \sum_{\mathbf{k}_1,\dots,\mathbf{k}_4\,:\, C_{h,j}(\mathbf{k}_i)>0} \hat{\psi}^{[h,j]+}_{\mathbf{k}_1,+} \hat{\psi}^{[h,j]-}_{\mathbf{k}_2,+} \hat{\psi}^{[h,j]+}_{\mathbf{k}_3,-} \hat{\psi}^{[h,j]-}_{\mathbf{k}_4,-}\, \delta(\mathbf{k}_1-\mathbf{k}_2+\mathbf{k}_3-\mathbf{k}_4). \tag{2.25}$$

∂_x and ∂_0 are defined in an obvious way, so that the second equality in (2.23) and (2.24) is satisfied; if $N=\infty$ they are simply the partial derivative with respect to x and x_0. Note that $\mathscr{L}\mathscr{V}^{(0)}=\mathscr{V}^{(0)}$, hence $l_0=\lambda$, $a_0=z_0=0$.

Analogously we write $\mathscr{B}^{(j)}=\mathscr{L}\mathscr{B}^{(j)}+\mathscr{R}\mathscr{B}^{(j)}$, $\mathscr{R}=1-\mathscr{L}$, according to the following definition. First of all, we put $\mathscr{L}W_R^{(j)}=W_R^{(j)}$. Let us consider now $\mathscr{B}_J^{(j)}(\sqrt{Z_j}\,\psi)$. It is easy to see that the field J is equivalent, from the point of view of dimensional considerations, to two ψ fields. Hence, the only terms which need a renormalized are those of second order in ψ, which are indeed marginal. We shall use for them the definition

$$\begin{aligned} \mathscr{B}_J^{(j,2)}(\sqrt{Z_j}\,\psi) &= \sum_{\omega,\tilde\omega} \int d\mathbf{x}\, d\mathbf{y}\, d\mathbf{z}\, B_{\omega,\tilde\omega}(\mathbf{x},\mathbf{y},\mathbf{z})\, J_{\mathbf{x},\omega} (\sqrt{Z_j}\,\psi^+_{\mathbf{y},\tilde\omega})(\sqrt{Z_j}\,\psi^-_{\mathbf{z},\tilde\omega}) \\ &= \sum_{\omega,\tilde\omega} \int \frac{d\mathbf{p}}{(2\pi)^2} \frac{d\mathbf{k}}{(2\pi)^2} \hat{B}_{\omega,\tilde\omega}(\mathbf{p},\mathbf{k})\, \hat{J}(\mathbf{p}) (\sqrt{Z_j}\,\hat{\psi}^+_{\mathbf{p}+\mathbf{k},\tilde\omega})(\sqrt{Z_j}\,\hat{\psi}^-_{\mathbf{k},\tilde\omega}). \end{aligned} \tag{2.26}$$

We regularize $\mathscr{B}_J^{(j,2)}(\sqrt{Z_j}\,\psi)$, in analogy to what we did for the effective potential, by decomposing it as the sum of $\mathscr{L}\mathscr{B}_J^{(j,2)}(\sqrt{Z_j}\,\psi)$ and

$\mathcal{R}\mathcal{B}_J^{(j,2)}(\sqrt{Z_j}\,\psi)$, where $\mathcal{L}$ is defined through its action on $\hat{B}_\omega(\mathbf{p},\mathbf{k})$ in the following way:

$$\mathcal{L}\hat{B}_{\omega,\tilde{\omega}}(\mathbf{p},\mathbf{k})=\tfrac{1}{4}\sum_{\eta,\eta'=\pm 1}\hat{B}_{\omega,\tilde{\omega}}(\bar{p}_\eta,\bar{k}_{\eta,\eta'}),\tag{2.27}$$

where $\bar{k}_{\eta,\eta'}$ was defined above and $\bar{p}_\eta=(0,2\pi\eta'/\beta)$. In the limit $L=\beta=\infty$ it reduces simply to $\mathcal{L}\hat{B}_{\omega,\tilde{\omega}}(\mathbf{p},\mathbf{k})=\hat{B}_{\omega,\tilde{\omega}}(0,0)$.

This definition apparently implies that we have to introduce two new renormalization constants. However, one can easily show that, in the limit $L,\beta\to\infty$, $\hat{B}_{\omega,-\omega}(0,0)=0$, while, at finite L and β, $\mathcal{L}B_{\omega,-\omega}$ behaves as an irrelevant term, see ref. 5.

The previous considerations imply that we can write

$$\mathcal{L}\mathcal{B}_J^{(j,2)}(\sqrt{Z_j}\,\psi)=\sum_\omega\frac{Z_j^{(2)}}{Z_j}\int d\mathbf{x}\,J_{\mathbf{x},\omega}(\sqrt{Z_j}\,\psi^+_{\mathbf{x},\omega})(\sqrt{Z_j}\,\psi^-_{\mathbf{x},\omega}),\tag{2.28}$$

which defines the renormalization constant $Z_j^{(2)}$.

Finally we have to define $\mathcal{L}$ for $\mathcal{B}_\phi^{(j)}(\sqrt{Z_j}\,\psi)$; we want to show that, by a suitable choice of the localization procedure, if $j\leqslant -1$, it can be written in the form

$$\begin{aligned}\mathcal{B}_\phi^{(j)}(\sqrt{Z_j}\,\psi)=\sum_\omega\sum_{i=j+1}^{0}\int d\mathbf{x}\,d\mathbf{y}\\ \cdot\Bigg[\phi^+_{\mathbf{x},\omega}g_\omega^{Q,(i)}(\mathbf{x}-\mathbf{y})\frac{\partial}{\partial\psi^+_{\mathbf{y}\omega}}\mathcal{V}^{(j)}(\sqrt{Z_j}\,\psi)\\ +\frac{\partial}{\partial\psi^-_{\mathbf{y},\omega}}\mathcal{V}^{(j)}(\sqrt{Z_j}\,\psi)\,g_\omega^{Q,(i)}(\mathbf{y}-\mathbf{x})\,\phi^-_{\mathbf{x},\omega}\Bigg]\\ +\sum_\omega\int\frac{d\mathbf{k}}{(2\pi)^2}\,[\hat{\psi}^{[h,j]+}_{\mathbf{k},\omega}\hat{Q}_\omega^{(j+1)}(\mathbf{k})\,\hat{\phi}^-_{\mathbf{k},\omega}+\hat{\phi}^+_{\mathbf{k},\omega}\hat{Q}_\omega^{(j+1)}(\mathbf{k})\,\hat{\psi}^{[h,j]-}_{\mathbf{k},\omega}],\end{aligned}\tag{2.29}$$

where $\hat{g}_\omega^{Q,(i)}(\mathbf{k})=\hat{g}_\omega^{(i)}(\mathbf{k})\,\hat{Q}_\omega^{(i)}(\mathbf{k})$ and $Q_\omega^{(j)}(\mathbf{k})$ is defined inductively by the relations

$$\hat{Q}_\omega^{(j)}(\mathbf{k})=\hat{Q}_\omega^{(j+1)}(\mathbf{k})-z_jZ_jD_\omega(\mathbf{k})\sum_{i=j+1}^{0}\hat{g}_\omega^{Q,(i)}(\mathbf{k}),\qquad \hat{Q}_\omega^{(0)}(\mathbf{k})=1.\tag{2.30}$$

The $\mathcal{L}$ operation for $\mathcal{B}_\phi^{(j)}$ is defined by decomposing $\mathcal{V}^{(j)}$ in the r.h.s. of (2.30) as $\mathcal{L}\mathcal{V}^{(j)}+\mathcal{R}\mathcal{V}^{(j)}$, $\mathcal{L}\mathcal{V}^{(j)}$ being defined by (2.22).

After writing $\mathscr{V}^{(j)} = \mathscr{L}\mathscr{V}^{(j)} + \mathscr{R}\mathscr{V}^{(j)}$ and $\mathscr{B}^{(j)} = \mathscr{L}\mathscr{B}^{(j)} + \mathscr{R}\mathscr{B}^{(j)}$, the next step is to *renormalize* the free measure $P_{\tilde{Z}_j, C_{h,j}}(d\psi^{[h,j]})$, by adding to it part of the r.h.s. of (2.22). We get

$$\int P_{\tilde{Z}_j, C_{h,j}}(d\psi^{[h,j]})\, e^{-\mathscr{V}^{(j)}(\sqrt{Z_j}\,\psi^{[h,j]}) + \mathscr{B}^{(j)}(\sqrt{Z_j}\,\psi^{[h,j]})}$$

$$= e^{-L\beta t_j} \int P_{\tilde{Z}_{j-1}, C_{h,j}}(d\psi^{[h,j]})\, e^{-\tilde{\mathscr{V}}^{(j)}(\sqrt{Z_j}\,\psi^{[h,j]}) + \tilde{\mathscr{B}}^{(j)}(\sqrt{Z_j}\,\psi^{[h,j]})}, \tag{2.31}$$

where

$$\tilde{Z}_{j-1}(\mathbf{k}) = Z_j(\mathbf{k})[1 + \chi_{h,j}(\mathbf{k})\, z_j], \tag{2.32}$$

$$\tilde{\mathscr{V}}^{(j)}(\sqrt{Z_j}\,\psi^{[h,j]}) = \mathscr{V}^{(j)}(\sqrt{Z_j}\,\psi^{[h,j]}) - z_j Z_j [F_\zeta^{[h,j]} + F_\alpha^{[h,j]}], \tag{2.33}$$

and the factor $\exp(-L\beta t_j)$ in (2.31) takes into account the different normalization of the two measures. Moreover

$$\tilde{\mathscr{B}}^{(j)}(\sqrt{Z_j}\,\psi^{[h,j]}) = \tilde{\mathscr{B}}_\phi^{(j)}(\sqrt{Z_j}\,\psi^{[h,j]}) + \mathscr{B}_J^{(j)}(\sqrt{Z_j}\,\psi^{[h,j]}) + W_R^{(j)}, \tag{2.34}$$

where $\tilde{\mathscr{B}}_\phi^{(j)}$ is obtained from $\mathscr{B}_\phi^{(j)}$ by inserting (2.33) in the second line of (2.29) and by absorbing the terms proportional to z_j in the terms in the third line of (2.29).

If $j > h$, the r.h.s of (2.31) can be written as

$$e^{-L\beta t_j} \int P_{\tilde{Z}_{j-1}, C_{h,j-1}}(d\psi^{[h,j-1]}) \int P_{Z_{j-1}, \tilde{f}_j^{-1}}(d\psi^{(j)})$$

$$e^{-\tilde{\mathscr{V}}^{(j)}(\sqrt{Z_j}\,[\psi^{[h,j-1]} + \psi^{(j)}]) + \tilde{\mathscr{B}}^{(j)}(\sqrt{Z_j}\,[\psi^{[h,j-1]} + \psi^{(j)}])}, \tag{2.35}$$

where $P_{Z_{j-1}, \tilde{f}_j^{-1}}(d\psi^{(j)})$ is the integration with propagator

$$\hat{g}_\omega^{(j)}(\mathbf{k}) = \frac{1}{Z_{j-1}} \frac{\tilde{f}_j(\mathbf{k})}{D_\omega(\mathbf{k})}, \tag{2.36}$$

with $\tilde{f}_j(\mathbf{k}) = f_j(\mathbf{k})\, Z_{j-1}[\tilde{Z}_{j-1}(\mathbf{k})]^{-1}$. Note that $\hat{g}_\omega^{(j)}(\mathbf{k})$ does not depend on the infrared cutoff for $j > h$ and that (even for $j = h$) $\hat{g}^{(j)}(\mathbf{k})$ is of size $Z_{j-1}^{-1}\gamma^{-j}$. Moreover the propagator $\hat{g}_\omega^{Q,(i)}(\mathbf{k})$ is equivalent to $\hat{g}_\omega^{(i)}(\mathbf{k})$, as concerns the dimensional bounds.

We now *rescale* the field so that

$$\begin{aligned} \tilde{\mathscr{V}}^{(j)}(\sqrt{Z_j}\,\psi^{[h,j]}) &= \hat{\mathscr{V}}^{(j)}(\sqrt{Z_{j-1}}\,\psi^{[h,j]}), \\ \tilde{\mathscr{B}}^{(j)}(\sqrt{Z_j}\,\psi^{[h,j]}) &= \hat{\mathscr{B}}^{(j)}(\sqrt{Z_{j-1}}\,\psi^{[h,j]}); \end{aligned} \tag{2.37}$$

it follows that

$$\mathscr{L}\hat{\mathscr{V}}^{(j)}(\psi^{[h,j]}) = \delta_j F_{\alpha}^{[h,j]} + \lambda_j F_{\lambda}^{[h,j]}, \tag{2.38}$$

where $\delta_j = (Z_j Z_{j-1}^{-1})(a_j - z_j)$ and $\lambda_j = (Z_j Z_{j-1}^{-1})^2 l_j$. If we now define

$$e^{-\mathscr{V}^{(j-1)}\sqrt{Z_j}(\psi^{[h,j-1]})+\mathscr{B}^{(j-1)}(\sqrt{Z_j}\psi^{[h,j-1]})-L\beta E_j}$$
$$= \int P_{Z_{j-1},\tilde{f}_j^{-1}}(d\psi^{(j)})\, e^{-\hat{\mathscr{V}}^{(j)}(\sqrt{Z_j}[\psi^{[h,j-1]}+\psi^{(j)}])+\hat{\mathscr{B}}^{(j)}(\sqrt{Z_j}[\psi^{[h,j-1]}+\psi^{(j)}])}, \tag{2.39}$$

it is easy to see that $\mathscr{V}^{(j-1)}$ and $\mathscr{B}^{(j-1)}$ are of the same form of $\mathscr{V}^{(j)}$ and $\mathscr{B}^{(j)}$ and that the procedure can be iterated. We call the set $\vec{v}_j = (\lambda_j, \delta_j)$ the *running coupling constants* on scale j. Note that the above procedure allows, in particular, to write the running coupling constants $\vec{v}_j$, $0 < j \leqslant h$, in terms of $\vec{v}_{j'}$, $0 \geqslant j' \geqslant j+1$:

$$\vec{v}_j = \vec{\beta}_j^{(h)}(\vec{v}_{j+1},..., \vec{v}_0), \qquad \vec{v}_0 = (\lambda, 0). \tag{2.40}$$

The function $\vec{\beta}_j^{(h)}(\vec{v}_{j+1},..., \vec{v}_0)$ is called the *Beta function*. By the remark above on the independence of scale j propagators of h for $j > h$, it is independent of h, for $j > h$.

At the end of the iterative integration procedure, we get

$$\mathscr{W}(\varphi, J) = -L\beta E_{L,\beta} + \sum_{m^\phi + n^J \geqslant 1} S^{(h)}_{2m^\phi, n^J}(\phi, J), \tag{2.41}$$

where $E_{L,\beta}$ is the *free energy* and $S^{(h)}_{2m^\phi, n^J}(\phi, J)$ are suitable functional, which can be expanded, as well as $E_{L,\beta}$, the effective potentials and the various terms in the r.h.s. of (2.17) and (2.16), in terms of trees (for an updated introduction to trees formalism see also ref. 11). This expansion, which is indeed a finite sum for finite values of N, L, β, is explained in detail in refs. 4 and 5, which we shall refer to often in the following.

Let us consider the family of all trees which can be constructed by joining a point r, the *root*, with an ordered set of $n \geqslant 1$ points, the *endpoints* of the *unlabeled tree*, so that r is not a branching point. Two unlabeled trees are identified if they can be superposed by a suitable continuous deformation, so that the endpoints with the same index coincide.

n will be called the *order* of the unlabeled tree and the branching points will be called the *non trivial vertices*. The unlabeled trees are partially ordered from the root to the endpoints in the natural way; we shall use the symbol $<$ to denote the partial order.

We shall consider also the *labelled trees* (to be called simply trees in the following), see Fig. 1; they are defined by associating some labels with the unlabeled trees, as explained in the following items.

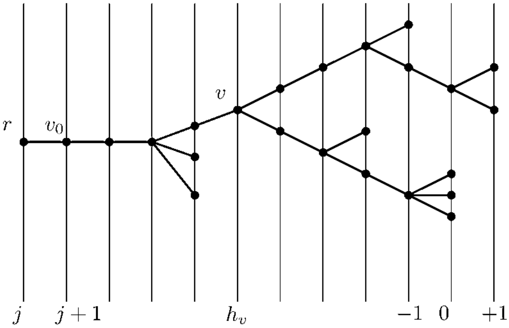

Fig. 1. A labelled tree.

(1) We associate a label $j \leqslant 0$ with the root and we denote $\mathcal{T}_{j,n}$ the corresponding set of labelled trees with n endpoints. Moreover, we introduce a family of vertical lines, labelled by an integer taking values in $[j, 1]$, and we represent any tree $\tau \in \mathcal{T}_{j,n}$ so that, if v is an endpoint or a non trivial vertex, it is contained in a vertical line with index $h_v > j$, to be called the *scale* of v, while the root is on the line with index j. There is the constraint that, if v is an endpoint, $h_v > j+1$.

The tree will intersect in general the vertical lines in set of points different from the root, the endpoints and the non trivial vertices; these points will be called *trivial vertices*. The set of the *vertices* of τ will be the union of the endpoints, the trivial vertices and the non trivial vertices. The definition of h_v is extended in an obvious way to the trivial vertices and the endpoints.

Note that, if v_1 and v_2 are two vertices and $v_1 < v_2$, then $h_{v_1} < h_{v_2}$. Moreover, there is only one vertex immediately following the root, which will be denoted v_0 and can not be an endpoint; its scale is $j+1$.

(2) There are two kind of endpoints, *normal* and *special*.

With each normal endpoint v of scale h_v we associate one of the two local terms contributing to $\mathcal{L}\hat{\mathcal{V}}^{(h_v)}(\psi^{[h, h_v-1]})$ in the r.h.s. of (2.38) and one space-time point $\mathbf{x}_v$. We shall say that the endpoint is of type δ or λ, with an obvious correspondence with the two terms. Note that there is no endpoint of type δ, if $h_v = +1$.

There are two types of special endpoints, to be called of type ϕ and J; the first one is associated with the terms in the third line of (2.30), the second one with the terms in the r.h.s. of (2.28). Given $v \in \tau$, we shall call n_v^ϕ and n_v^J the number of endpoints of type ϕ and J following v in the tree, while n_v will denote the number of normal endpoints following v. Analogously, given τ, we shall call n_τ^ϕ and n_τ^J the number of endpoint of type ϕ and J, while n_τ will denote the number of normal endpoints. Finally, $\mathcal{T}_{j,n,n^\phi,n^J}$ will denote the set of trees belonging to $\mathcal{T}_{j,n}$ with n normal endpoints, n^ϕ endpoints of type ϕ and n^J endpoints of type J. Given a vertex v, which is not an endpoint, $\mathbf{x}_v$ will denote the family of all space-time points associated with one of the endpoints following v.

(3) There is an important constraint on the scale indices of the endpoints. In fact, if v is an endpoint normal or of type J, $h_v = h_{v'} + 1$, if v' is the non trivial vertex immediately preceding v. This constraint takes into account the fact that at least one of the ψ fields associated with an endpoint normal or of type J has to be contracted in a propagator of scale $h_{v'}$, as a consequence of our definitions.

On the contrary, if v is an endpoint of type ϕ, we shall only impose the condition that $h_v \geqslant h_{v'} + 1$. In this case the only ψ field associated with v is contracted in a propagator of scale $h_v - 1$, instead of $h_{v'}$.

(4) If v is not an endpoint, the *cluster* L_v with frequency h_v is the set of endpoints following the vertex v; if v is an endpoint, it is itself a (*trivial*) cluster. The tree provides an organization of endpoints into a hierarchy of clusters.

(5) We associate with any vertex v of the tree a set P_v, the *external fields* of v. The set P_v includes both the field variables of type ψ which belong to one of the endpoints following v and are not yet contracted at scale h_v (in the iterative integration procedure), to be called *normal external fields*, and those which belong to an endpoint normal or of type J and are contracted with a field variable belonging to an endpoint $\tilde{v}$ of type ϕ through a propagator $g^{Q,(h_{\tilde{v}}-1)}$, to be called *special external fields* of v.

These subsets must satisfy various constraints. First of all, if v is not an endpoint and $v_1,..., v_{s_v}$ are the s_v vertices immediately following it, then $P_v \subset \bigcup_i P_{v_i}$. We shall denote Q_{v_i} the intersection of P_v and P_{v_i}; this definition implies that $P_v = \bigcup_i Q_{v_i}$. The subsets $P_{v_i} \setminus Q_{v_i}$, whose union will be made, by definition, of the *internal fields* of v, have to be non empty, if $s_v > 1$, that is if v is a non trivial vertex.

Moreover, if the set P_{v_0} contains only special external fields, that is if $|P_{v_0}| = n^\phi$, and $\tilde{v}_0$ is the vertex immediately following v_0, then $|P_{v_0}| < |P_{\tilde{v}_0}|$.

We can write

$$S^{(h)}_{2m^\phi, n^J}(\phi, J) = \sum_{n=0}^{\infty} \sum_{j_0=h-1}^{-1} \sum_{\substack{\tau \in \mathscr{T}_{j_0, n, 2m^\phi, n^J} \\ |P_{v_0}| = 2m^\phi}} \sum_{\underline{\omega}} \int d\underline{\mathbf{x}}$$

$$\times \prod_{i=1}^{2m^\phi} \phi^{\sigma_i}_{\mathbf{x}_i, \omega_i} \prod_{r=1}^{n^J} J_{\mathbf{x}_{2m^\phi+r}, \omega_{2m^\phi+r}} S_{2m^\phi, n^J, \tau, \underline{\omega}}(\underline{\mathbf{x}}), \tag{2.42}$$

where $\underline{\omega} = \underline{\omega} = \{\omega_1,..., \omega_{2m^\phi+n^J}\}$, $\underline{\mathbf{x}} = \{\mathbf{x}_1,..., \mathbf{x}_{2m^\phi+n^J}\}$ and $\sigma_i = +$ if i is odd, $\sigma_i = -$ if i is even. Moreover, the kernels $S_{2m^\phi, n^J, \tau, \underline{\omega}}(\underline{\mathbf{x}})$ are suitable functions, whose explicit expression can be found in ref. 5 in the case $m^\phi = 0$ and can be easily extended to the general case; the case $m^\phi = 1$, $n^J = 0$ is considered in detail in ref. 4. We shall not report it here, but we only remark that the kernels satisfy the following dimensional bound:

$$\int d\underline{\mathbf{x}}\, |S_{2m^\phi, n^J, \tau, \underline{\omega}}(\underline{\mathbf{x}})| \leqslant L\beta(C\bar{\varepsilon})^n \gamma^{-j_0(-2+m^\phi+n^J)} \prod_{i=1}^{2m^\phi} \frac{\gamma^{-h_i}}{(Z_{h_i})^{1/2}} \cdot \prod_{r=1}^{n^J} \frac{Z^{(2)}_{\bar{h}_r}}{Z_{\bar{h}_r}} \prod_{\mathrm{v\ not\ e.p}} \left(\frac{Z_{h_v}}{Z_{h_v-1}}\right)^{|P_v|/2} \gamma^{-d_v}, \tag{2.43}$$

where $\bar{\varepsilon} = \max_{0 \geqslant k \geqslant j} |\vec{v}_k|$, h_i is the scale of the propagator linking the ith endpoint of type ϕ to the tree, $\bar{h}_r$ is the scale of the rth endpoint of type J and

$$d_v = -2 + |P_v|/2 + n^J_v + \tilde{z}(P_v), \tag{2.44}$$

with

$$\tilde{z}(P_v) = \begin{cases} z(P_v) & \text{if} \quad n^\phi_v \leqslant 1, \quad n^J_v = 0, \\ 1 & \text{if} \quad n^\phi_v = 0, \quad n^J_v = 1, \quad |P_v| = 2, \\ 0 & \text{otherwise} \end{cases} \tag{2.45}$$

and $z(P_v) = 1$ if $|P_v| = 4$, $z(P_v) = 2$ if $|P_v| = 2$ and zero otherwise.

From the above bound we can easily get the asymptotic behaviour of the Schwinger functions we are interested in. In fact the Schwinger functions are simply related to the kernels of the functionals $S^{(h)}_{2m^\phi, n^J}(\phi, J)$ and (2.43) allows to get an expansion for them. For example, $G^2_\omega(\mathbf{x}_1, \mathbf{x}_2)$ is equal to the sum over the terms in the r.h.s. of (2.43) with $m^\phi = 1$, $n^J = 0$ and $\underline{\omega} = (\omega, \omega)$, while $G^{2,1}_\omega(\mathbf{x}; \mathbf{y}, \mathbf{z})$ is obtained by selecting the terms with $m^\phi = 1$, $n^J = 1$ and $\underline{\omega} = (\omega, \omega, \omega)$. Hence, the bound (2.43) is sufficient to get a bound for the Schwinger functions Fourier transforms, if $\bar{\varepsilon}$ is small enough, because, by translation invariance, the Fourier transform of $S_{2m^\phi, n^J, \tau, \underline{\omega}}(\underline{\mathbf{x}})$ is bounded by $(L\beta)^{-1} \int d\underline{\mathbf{x}}\, |S_{2m^\phi, n^J, \tau, \underline{\omega}}(\underline{\mathbf{x}})|$. We only have to sum over τ the r.h.s. of (2.43) (without the $L\beta$ factor), by using the techniques described in detail in ref. 5. The main point is to control the sums over the sets P_v and the scale indices h_v, for fixed values of the external propagators scale indices h_i, which are determined up to one unit by the external momenta. Hence, if all the "vertex dimensions" d_v were greater than 0, one would get a dimensional bound of the type

$$(C\bar{\varepsilon})^{\bar{n}} \sum_{j_0=h}^{\bar{h}} \gamma^{-j_0(-2+m^\phi+n^J)} \prod_{i=1}^{2m^\phi} \frac{\gamma^{-h_i}}{(Z_{h_i})^{1/2}} \prod_{r=1}^{n^J} \frac{Z^{(2)}_{\bar{h}_r}}{Z_{\bar{h}_r}}, \tag{2.46}$$

where $\bar{n}$ is the minimal order in λ of the graphs contributing to the Schwinger function and $\bar{h}$ is an upper bound on the scale of the tree lower vertex v_0, which depends on the external momenta.

However, it is not true that, given τ, $d_v > 0$ for all non trivial $v \in \tau$; in fact $d_v = 0$, if $|P_v| = 2$ and $n_v^\phi = n_v^J = 1$ or $n_v^\phi = 2, n_v^J = 0$. This implies that the sum over the scale indices of some special paths on the tree can produce a result different from the "trivial one," leading to (2.46). Hence, in order to get the right bound, one has to analyse case by case the constraints on the endpoint scale indices, related to the support properties of the single scale propagators and the fact that the ϕ and J momenta are fixed.

3. VANISHING OF LUTTINGER BETA FUNCTION

3.1. Vanishing of Beta Function and Smallness of Running Coupling Constants

In the previous section we have defined, for each fixed $h<0$, an expansion of the Schwinger functions for the model with infrared cutoff γ^h, in terms of the running coupling constants $\{\vec{v}_j\}_{h \leqslant j \leqslant 0}$; if $\bar{\varepsilon} = \max_{h \leqslant j \leqslant 0} |\vec{v}_j|$ is small enough, such expansion is convergent. Moreover, it is easy to see that all results are true even if we add to the interaction (2.3) a term $\delta_0 F_\alpha^{[h,0]}$ (see (2.23)), with δ_0 of order λ. In fact, we never used the fact that $\alpha_0 = \delta_0 = 0$ in an essential way and the introduction of this term has the physical meaning of a small change in the free Fermi velocity (which we put equal to 1, for simplicity).

The fact that $\bar{\varepsilon} = \max_{h \leqslant j \leqslant 0} |\vec{v}_j|$ can be chosen small with λ is a consequence of the following remarkable property, to be proved in Section 3.3.

Theorem 3.1. There are $\bar{\varepsilon}_0 > 0$ and $\eta' < 1$, such that, for any $j<0$, if $|\vec{v}| \leqslant \bar{\varepsilon}_0$:

$$|\beta_{j,\lambda}(\vec{v},..., \vec{v})| \leqslant C\, |\vec{v}|^2\, \gamma^{\eta' j}, \quad |\beta_{j,\delta}(\vec{v},..., \vec{v})| \leqslant C\, |\vec{v}|^2\, \gamma^{\eta' j}, \quad 0<\eta'<1, \tag{3.1}$$

where $\vec{\beta}_j = (\beta_{j,\lambda}, \beta_{j,\delta})$ is defined as in (2.41), with $h=-\infty$.

The above property is usually called "vanishing of the Beta function," and it is an highly non trivial statement. In fact each order of the expansion for $\beta_j(\vec{v},..., \vec{v})$ is given by a sum of *Feynman graphs* having a non-vanishing limit as $j \to -\infty$. What (3.1) says is that there are cancellations among Feynman graphs so that the sum is $O(\gamma^{\eta' j})$. Of course it is easy to check this cancellation at the second order by an explicit computation; at the third order, to see the cancellation is already quite cumbersome and we think that it is essentially impossible to check it at every order in the expansion. Hence we prove (3.1) by using the exact solution of the Luttinger model, following the strategy first proposed in ref. 2. As we said in the introduction, the interest of (3.1) is that it can be used in the analysis of many different models, like spin chain or coupled Ising models, for which an exact solution is not available, at least for the correlation functions.

An immediate consequence of (3.1) is the following lemma, see also ref. 5.

Lemma 3.1. If (3.1) holds and λ is small enough, then, for any infrared cutoff scale h and any $j > h$, $\bar{\varepsilon}_j \equiv \max_{j \leqslant i \leqslant 0} |\vec{v}_i| \leqslant C\, |\lambda|$.

Proof. Note first that, by the compact support properties of the propagator, $\vec{\beta}_j = \vec{\beta}_j^{(h)}$, for any $h < j$. Hence, for any fixed j, by taking the infrared cutoff scale h smaller than $j-1$, we can write

$$\vec{v}_{j-1} = \vec{v}_j + \vec{\beta}_j(\vec{v}_j,..., \vec{v}_j) + \sum_{k=j+1}^{0} \vec{D}_{j,k}, \tag{3.2}$$

where

$$\vec{D}_{j,k} = \vec{\beta}_j(\vec{v}_j,..., \vec{v}_j, \vec{v}_k, \vec{v}_{k+1},..., \vec{v}_0) - \vec{\beta}_h(\vec{v}_j,..., \vec{v}_j, \vec{v}_j, \vec{v}_{k+1},..., \vec{v}_0). \tag{3.3}$$

On the other hand, it is easy to see that $\vec{D}_{j,k}$ admits a tree expansion similar to that of $\vec{\beta}_j(\vec{v}_j,..., \vec{v}_1)$, with the property that all trees giving a non zero contribution must have an endpoint of scale $k+1$, associated with a difference $\lambda_k - \lambda_j$ or $\delta_k - \delta_j$. Moreover, it is easy to see that our expansion has the property that the trees with root of scale j, containing an endpoint of scale i, are damped by a factor $\gamma^{-\eta(i-j)}$, for some positive constant η; hence

$$|\vec{D}_{j,k}| \leqslant C \bar{\varepsilon}_j \gamma^{-\eta(k-j)}\, |\vec{v}_k - \vec{v}_j|. \tag{3.4}$$

We want now to show that there exists a constant c_0, such that, uniformly in j,

$$|\vec{v}_{k-1} - \vec{v}_k| \leqslant c_0\, |\lambda|^{3/2}\, \gamma^{\theta k}, \qquad j < k \leqslant 1. \tag{3.5}$$

with $\theta = \min\{\eta/2, \eta'\}$. In fact (3.5) is certainly verified for $k=1$ and, by using (3.1), (3.2), and (3.5),

$$|\vec{v}_{j-1} - \vec{v}_j| \leqslant C\bar{\varepsilon}_j^2 \gamma^{\eta' j} + C c_0 \bar{\varepsilon}_j^{5/2} \sum_{k=j+1}^{1} \gamma^{-\eta(k-j)} \sum_{i=j+1}^{k} \gamma^{\theta i} \tag{3.6}$$

which immediately implies (3.5) with $j \to j-1$, together with the condition $\bar{\varepsilon}_j \leqslant c_1 |\lambda|$, for some constant c_1, independent of j. ▮

3.2. Comparison with the Luttinger Model Beta Function

Let us consider the Luttinger model with hamiltonian

$$\begin{gathered} H = H_0 + V, \qquad H_0 = \sum_{\omega = \pm 1} i\omega(1+\delta) \int_0^L dx : \psi^+_{\omega, x}\, \partial_x \psi^-_{\omega, x} :, \\ V = \lambda \int_0^L dx \int_0^L dy\, v(x-y) : \psi^+_{+1, x} \psi^-_{+1, x} :: \psi^+_{-1, y} \psi^-_{-1, y} :, \end{gathered} \tag{3.7}$$

where :: denotes the Wick ordering and $v(x)$ is a smooth function of fast decay. Note that we have eliminated the dependence on the Fermi momentum p_F by a trivial redefinition of the fermionic fields and we have put the Fermi velocity equal to $1+\delta$. Since the Fermi velocity has to be positive, we shall suppose that $|\delta| \leqslant 1/2$.

The crucial property of such model is that it is exactly soluble (as it was shown by Mattis and Lieb[20]) and its Schwinger functions can be computed,[3] in the limit $\beta = \infty$. In particular this is true for the two and four point Schwinger functions; from Eq. (2.4) of ref. 3 (slightly modified in order to take into account that the Fermi velocity is $1+\delta$ instead of 1) it follows that, for any finite L and $\beta = \infty$,

$$G^{4, L}_+(\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \mathbf{x}_4) = G^{2, L}_+(\mathbf{x}_1 - \mathbf{x}_2)\, G^{2, L}_-(\mathbf{x}_3 - \mathbf{x}_4)[e^{A(\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \mathbf{x}_4)} - 1], \tag{3.8}$$

where $G^{4, L}_\omega$ and $G^{2, L}_\omega$ are defined analogously to (2.11) and (2.10), respectively, and

$$\begin{gathered} A(\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \mathbf{x}_4) = F(\mathbf{x}_1 - \mathbf{x}_4) + F(\mathbf{x}_2 - \mathbf{x}_3) - F(\mathbf{x}_1 - \mathbf{x}_3) - F(\mathbf{x}_2 - \mathbf{x}_4), \\ F(\mathbf{x}) = \frac{2\pi}{L} \sum_{p>0} \frac{s(p)\, c(p)}{p} (1 - e^{-p|x_0|(1+\delta)\mu(p)} \cos px), \end{gathered} \tag{3.9}$$

with $s(p) = \sinh\phi(p)$, $c(p) = \cosh\phi(p)$, $\mu(p) = e^{-2\phi(p)}$, $\tanh 2\phi(p) = -\lambda\hat{v}(p)[\lambda\hat{v}(p) + 4\pi(1+\delta)]^{-1}$ and $p = 2m\pi/L$, m integer. Note that $(1+\delta)\,\mu(p) \geqslant a$, for some constant $a > 0$.

We consider now $G_+^{4,L}$ for values $\bar{\mathbf{x}}_i$, such that $|\bar{\mathbf{x}}_i - \bar{\mathbf{x}}_j| \leqslant L/2$ for all couples (i, j); a convenient choice is

$$\bar{\mathbf{x}}_1 = (r, r), \qquad \bar{\mathbf{x}}_2 = (-r, r), \qquad \bar{\mathbf{x}}_3 = (-r, -r), \qquad \bar{\mathbf{x}}_4 = (r, -r), \tag{3.10}$$

where $0 \leqslant r \leqslant L/4$. One can immediately check that

$$A(\bar{\mathbf{x}}_1, \bar{\mathbf{x}}_2, \bar{\mathbf{x}}_3, \bar{\mathbf{x}}_4) = \frac{4\pi}{L} \sum_{p>0} \frac{s(p)\,c(p)}{p} \left[e^{-2\sqrt{2}\,pr(1+\delta)\,\mu(p)} \cos(2\sqrt{2}\,pr) - e^{-2pr(1+\delta)\,\mu(p)}\right]. \tag{3.11}$$

It is then very easy to show that

$$\left| \frac{G_+^{4,L}(\bar{\mathbf{x}}_1, \bar{\mathbf{x}}_2, \bar{\mathbf{x}}_3, \bar{\mathbf{x}}_4)}{G_+^{2,L}(\bar{\mathbf{x}}_1 - \bar{\mathbf{x}}_2)\, G_-^{2,L}(\bar{\mathbf{x}}_3 - \bar{\mathbf{x}}_4)} \right| \leqslant c_1\, |\lambda| \tag{3.12}$$

for some constant c_1, independent of L and δ.

On the other hand we can compute the Schwinger functions of the Luttinger model also by a Renormalization Group expansion, by taking $L \ll \beta$ (and at the end the limit $\beta \to \infty$ is taken). We start from a generating functional like (2.8), with

$$\mathscr{V}(\psi) = \lambda \int d\mathbf{x}\, v(x-y)\, \delta(x_0 - y_0) : \psi^+_{\mathbf{x},+} \psi^-_{\mathbf{x},+} :: \psi^+_{\mathbf{y},-} \psi^-_{\mathbf{y},-} : + \\
+ \sum_{\omega} i\omega\delta \int d\mathbf{x} : \psi^+_{\mathbf{x},\omega}\, \partial_x \psi^-_{\mathbf{x},\omega} : \tag{3.13}$$

and

$$P(d\psi) = \mathscr{N}^{-1} \mathscr{D}\psi \exp\left\{ -\frac{1}{L\beta} \sum_{\omega=\pm 1} \sum_{\mathbf{k}} (-ik_0 + \omega k)\, \hat{\psi}^+_{\mathbf{k},\omega} \hat{\psi}^-_{\mathbf{k},\omega} \right\}. \tag{3.14}$$

We can write

$$P(d\psi) = P_L(d\psi^{i.r})\, P_l(d\psi^{u.v.}), \tag{3.15}$$

where $P_L(d\psi^{i.r})$ is given by (2.4) with $h = h_L$, h_L being the smallest h such that πL^{-1} is in the support of f_h, while $P_l(d\psi^{u.v.})$ has the same expression, with $1 - C_{h_L,0}(\mathbf{k})$ in place of $C_{h_L,0}(\mathbf{k})$. The ultraviolet problem of the Luttinger model was discussed in ref. 9, with a different choice of the cutoff function, but their results hold also with the present choice. The analysis of ref. 9 implies that, if $(\psi, \phi)\sum_\omega \int d\mathbf{x}[\phi^+_{\mathbf{x},\omega}\psi^-_{\mathbf{x},\omega}+\psi^+_{\mathbf{x},\omega}\phi^-_{\mathbf{x},\omega}]$,

$$\int P_l(d\psi^{u.v.})\, e^{\mathscr{V}(\psi)+(\psi,\phi)} = e^{E_0+\mathscr{V}^{(0)}(\psi^{i.r})+\mathscr{B}^0(\psi^{i.r},\phi)}, \tag{3.16}$$

where E_0 is an analytic function of λ and δ, $V^{(0)}(\psi^{i.r})$ can be written as in (2.16), with kernels analytic in λ and δ, and $\mathscr{B}^0(\psi,\phi)$ has an expression like (2.17) (with $J=0$) and has similar analyticity properties. Moreover, all the kernels have fast decaying properties on scale 0 (space-time distances of order 1).

Let us now observe that we can write $\mathscr{V}^{(0)}(\psi^{i.r})+\mathscr{B}^0(\psi^{i.r},\phi)$ as the sum of a local part plus a remainder, which contains all possible "irrelevant" terms. Since the local part has the same structure as the local part of the model studied in Section 2, with only different values of the running coupling constants on scale 0, and the irrelevant terms are of the same type of those produced in Section 2 by the first infrared integration, it is clear that we can repeat the analysis done for the model (2.8) also for the Luttinger model (note that the analysis was done with L, β finite; this will play a crucial role in the following). Moreover, the arguments used in the proof of Lemma 4.5 of ref. 5 allow us to prove, without any further subtle problem, that adding irrelevant terms to the effective interaction on scale 0 has an exponentially small effect on scale j, for $j\to-\infty$. Hence, if we call $\vec{v}^L_j=(\lambda^L_j,\delta^L_j)$, $0\geqslant j\geqslant h_L$, the running coupling constants in the Luttinger model at volume L, with $\vec{v}^L_0=(\lambda,\delta)$, and $\vec{\beta}^l_j(\vec{v}^L_j,\dots,\vec{v}^L_0)$ the Luttinger model Beta function, the following Lemma can be proved.

Lemma 3.2. There are $\varepsilon_0>0$ and $\eta'<1$ (independent of L), such that, given $j\leqslant 0$, $\vec{\beta}^l_j(\vec{v}^L_j,\dots,\vec{v}^L_0)$ is an analytic function of his arguments in the region $\bar{\varepsilon}_L=\max_{0\geqslant j\geqslant h_L}|\vec{v}^L_j|\leqslant\varepsilon_0$, for some small ε_0. Moreover, if $\bar{\varepsilon}_j=\max_{0\geqslant i\geqslant j}|\vec{v}^L_i|$

$$|\vec{\beta}_j(\vec{v}^L_j,\dots,\vec{v}^L_0)-\vec{\beta}^l_j(\vec{v}^L_j,\dots,\vec{v}^L_0)|\leqslant C\bar{\varepsilon}^2_j\gamma^{\eta j}. \tag{3.17}$$

3.3. Proof of Theorem 3.1

Lemma 3.2 implies that, if we prove (3.1) for the Luttinger model Beta function, the same property holds for the model (2.8). In order to prove the vanishing of the Beta function in the Luttinger model, we will use the following bound, obtained through our Renormalization Group expansion.

Lemma 3.3. Suppose that $\bar{\mathbf{x}}_1, \bar{\mathbf{x}}_2, \bar{\mathbf{x}}_3, \bar{\mathbf{x}}_4$ are chosen as in (3.10), with r such that $\gamma^{-h_r} = r \leqslant L/4$ and $h_r - h_L = \bar{m}$, and that $\bar{\varepsilon}_L \leqslant \varepsilon_0$. Then, if $\bar{\lambda}_L \equiv \max_{1 \geqslant j \geqslant h_L} |\lambda_j^L|$, there exists a constant $a_{\bar{m}} > 0$, independent of L, such that

$$\left| \frac{G_+^{4, L}(\bar{\mathbf{x}}_1, \bar{\mathbf{x}}_2, \bar{\mathbf{x}}_3, \bar{\mathbf{x}}_4)}{G_+^{2, L}(\bar{\mathbf{x}}_1 - \bar{\mathbf{x}}_2)\, G_-^{2, L}(\bar{\mathbf{x}}_3 - \bar{\mathbf{x}}_4)} - \lambda_{h_r}^L a_{L, \bar{m}} \right| \leqslant c_0 \bar{\lambda}_L \bar{\varepsilon}_L, \tag{3.18}$$

for some constants c_0 and $a_{L, \bar{m}}$, with $|a_{L, \bar{m}}| \geqslant a_{\bar{m}}$.

Proof. To start with, we want to prove that, if $h_r = h_L + \bar{m}$,

$$\left| G_+^{4, L}(\bar{\mathbf{x}}_1, \bar{\mathbf{x}}_2, \bar{\mathbf{x}}_3, \bar{\mathbf{x}}_4) - \lambda_{h_r}^L c_{L, \bar{m}} \frac{\gamma^{2h_r}}{Z_{h_r}^2} \right| \leqslant \bar{c}_0 \bar{\lambda}_L \bar{\varepsilon}_L \frac{\gamma^{2h_r}}{Z_{h_r}^2}, \tag{3.19}$$

where $c_{L, \bar{m}}$ is a suitable constant, bounded away from 0, uniformly in L, for any fixed $\bar{m}$. We shall use the expansion described in Section 2, for which the bound (2.43) was found. We have to modify such bound by taking into account the fact that there is no integrations over the coordinates and that all differences of the coordinates are of order γ^{-h_r}.

By using (2.42), we can write

$$G_+^{4, L}(\bar{\mathbf{x}}_1, \bar{\mathbf{x}}_2, \bar{\mathbf{x}}_3, \bar{\mathbf{x}}_4) = \sum_{n=1}^{\infty} \sum_{j_0 = h_L - 1}^{-1} \sum_{\substack{\tau \in \mathscr{T}_{j_0, n, 4, 0} \\ |P_{v_0}| = 4}} G_{4, \tau}(\bar{\mathbf{x}}_1, \bar{\mathbf{x}}_2, \bar{\mathbf{x}}_3, \bar{\mathbf{x}}_4)$$
$$+ G_+^{4, L, (uv)}(\bar{\mathbf{x}}_1, \bar{\mathbf{x}}_2, \bar{\mathbf{x}}_3, \bar{\mathbf{x}}_4), \tag{3.20}$$

where $G_{4, \tau} \equiv S_{4, 0, \tau, \{+, +, -, -\}}$ and $G_+^{4, L, (uv)}$ is the contribution of the ultra-violet scales (it is the kernel of a term of forth order in ϕ contributing to $\mathscr{B}^0(0, \phi)$, see (3.16), which gives a negligible contribution for $r \to \infty$. We shall divide the trees contributing to the first term in the r.h.s. of (3.20) in two families, defined in terms of some properties of the four special end-points of type ϕ associated with the four points $\bar{\mathbf{x}}_i$.

Let us consider first the family $\mathscr{T}_n^{(1)}$ of trees with n endpoints sharing the following properties.

(1) If v_i, $i = 1,..., 4$, are the four special endpoints, there are a permutation (a, b, c, d) of $(1, 2, 3, 4)$ and two vertices v_{ab} and v_{cd}, such that $v_{ab} < v_a, v_b$ and $v_{cd} < v_c, v_d$;

(2) if $v < v_a, v_b$ ($v < v_c, v_d$) then $v \leqslant v_{ab}$ ($v \leqslant v_{cd}$);

(3) if $v_{ab} \neq v_{cd}$, then the subtrees with root in the vertices (possibly coinciding) immediately preceding v_{ab} and v_{cd} are disjoint.

These conditions essentially imply that v_{ab} is the higher vertex such that $\mathbf{x}_{v_{ab}}$ contains both $\mathbf{x}_a$ and $\mathbf{x}_b$, but does not contain $\mathbf{x}_c$ and $\mathbf{x}_d$; a similar condition is valid for v_{cd}. Moreover, there is a vertex $\bar{v}$, which is the higher one preceding both v_{ab} and v_{cd}; note that, if $v_{ab} = v_{cd}$, then $\bar{v} = v_{ab} = v_{cd}$.

By using the notation of ref. 5, Section 5, we call $\tilde{T}_{v_{ab}} = \bigcup_{v \geqslant v_{ab}} \tilde{T}_v$ the subtree of the tree graph connecting the points in $\mathbf{x}_{v_{ab}}$; recall that $\mathbf{x}_v$ is the set of all vertices associated with the endpoints following v and that there is a propagator of scale h_v associated with any line $l \in \tilde{T}_v$. Proceeding as in ref. 5, Section 5.9, we can extract from the propagators in $\tilde{T}_{v_{ab}}$ a factor smaller than $nC_N(1+\gamma^{h_{v_{ab}}} |\mathbf{x}_a - \mathbf{x}_b|)^{-N}$, if n is the number of endpoints, since there is in $\tilde{T}_{v_{ab}}$ a path connecting $\bar{\mathbf{x}}_a$ with $\bar{\mathbf{x}}_b$.

Note that here we are using the condition $r \leqslant L/4$, in order to be able to substitute the distance on the torus (recall that we work with antiperiodic boundary conditions) with the Euclidean distance.

In a similar way, we can extract from the propagators in $\tilde{T}_{v_{cd}}$ a factor smaller $nC_N(1+\gamma^{h_{v_{cd}}} |\mathbf{x}_c - \mathbf{x}_d|)^{-N}$. An analogous argument can be applied to the the vertex $\bar{v}$, by choosing any couple of special endpoints (all distances between the $\bar{\mathbf{x}}_i$ are of the same order γ^{-h_r}); for example, we can extract a factor $nC_N(1+\gamma^{h_{\bar{v}}} |\mathbf{x}_a - \mathbf{x}_c|)^{-N}$. If we take $N = 1$, the product of these three factors is bounded by $C\gamma^{-(h_{v_{ab}} - h_r)}\gamma^{-(h_{v_{cd}} - h_r)}\gamma^{-(h_{\bar{v}} - h_r)}$.

The bound of $G_{4,\tau}(\bar{\mathbf{x}}_1, \bar{\mathbf{x}}_2, \bar{\mathbf{x}}_3, \bar{\mathbf{x}}_4)$ will differ from the r.h.s. of (2.43), because of this factor and because we have to do three integration less (there are four fixed points, instead of one). Since the integrations leading to (2.43) are done by using the decaying properties of the propagators associated with the lines of $\tilde{T}_{v_0}$, it is easy to see that we can choose the "missing integrations," so that we gain a factor $\gamma^{2h_{v_{ab}} + 2h_{v_{cd}} + 2h_{\bar{v}}}$. It follows that, if $\tau \in \mathscr{T}_n^{(1)}$,

$$|G_{4,\tau}(\bar{\mathbf{x}}_1, \bar{\mathbf{x}}_2, \bar{\mathbf{x}}_3, \bar{\mathbf{x}}_4)| \leqslant C\gamma^{3\bar{m}}\bar{\lambda}_L(C\bar{\varepsilon}_L)^{n-1} \prod_{i=1}^{4} \frac{\gamma^{-h_i}}{(Z_{h_i})^{1/2}} \prod_{\text{v not e.p}} \gamma^{-d_v}$$
$$\cdot \gamma^{2h_{v_{ab}} + 2h_{v_{cd}} + 2h_{\bar{v}}}\gamma^{-(h_{v_{ab}} - h_L)}\gamma^{-(h_{v_{cd}} - h_L)}\gamma^{-(h_{\bar{v}} - h_L)}, \qquad (3.21)$$

where we used also the fact that $Z_j/Z_{j-1} < 1$ (see ref. 5) and that there is at least an endpoint of type λ.

Note that, by (2.44), d_v can be equal to 0 for all vertices belonging to the paths $\mathscr{C}_{ab}$ and $\mathscr{C}_{cd}$, which connect $\bar{v}$ with v_{ab} and v_{cd}, respectively, while $d_v > 0$ in all other vertices. However,

$$\gamma^{-(h_{v_{ab}} - h_L)}\gamma^{-(h_{v_{cd}} - h_L)} \prod_{\text{v not e.p}} \gamma^{-d_v} = \gamma^{-2(h_{\bar{v}} - h_L)} \prod_{\text{v not e.p}} \gamma^{-\tilde{d}_v}, \qquad (3.22)$$

with $\tilde{d}_v > 0$ for all $v \in \tau$. Moreover, if v_i' is the vertex immediately preceding v_i (see item 3) after Fig. 1, we have

$$\sum_{\substack{h_i \geqslant h_{v_i'} \\ i=1,\dots,4}} \gamma^{2h_{v_{ab}}+2h_{v_{cd}}} \prod_{i=1}^{4} \frac{\gamma^{-h_i}}{(Z_{h_i})^{1/2}} \leqslant \frac{C}{Z_{h_{\bar{v}}}^2} \prod_{i=1}^{4} \left(\frac{Z_{h_{\bar{v}}}}{Z_{h_{v_i'}}} \right)^{1/2}. \tag{3.23}$$

By using that $Z_i/Z_j \leqslant \gamma^{c\bar{\varepsilon}_L^2(j-i)}$, if $j > i$ (see refs. 5), it follows that

$$\begin{aligned} &\sum_{\tau \in \mathscr{T}_n^{(1)}} |G_{4,\tau}(\bar{\mathbf{x}}_1, \bar{\mathbf{x}}_2, \bar{\mathbf{x}}_3, \bar{\mathbf{x}}_4)| \\ &\quad \leqslant \sum_{\bar{h}=h_L}^{0} \sum_{\tau \in \mathscr{T}_n^{(1)} : h_{\bar{v}} = \bar{h}} |G_{4,\tau}(\bar{\mathbf{x}}_1, \bar{\mathbf{x}}_2, \bar{\mathbf{x}}_3, \bar{\mathbf{x}}_4)| \\ &\quad \leqslant C\bar{\lambda}_L (C\bar{\varepsilon}_L)^{n-1} \sum_{\bar{h}=h_L}^{0} \frac{\gamma^{2\bar{h}} \gamma^{-3(\bar{h}-h_L)}}{Z_{h_{\bar{v}}}^2} \leqslant C\bar{\lambda}_L (C\bar{\varepsilon}_L)^{n-1} \frac{\gamma^{2h_r}}{Z_{h_r}^2}. \end{aligned} \tag{3.24}$$

A similar bound can be found for the second family $\mathscr{T}_n^{(2)} = \mathscr{T}_n \backslash \mathscr{T}_n^{(1)}$ of trees contributing to the first term in the r.h.s. of (3.20). If $\tau \in \mathscr{T}_n^{(2)}$, there is a vertex v_{ab}, which is the first vertex $v \in \tau$ such that the two points $\mathbf{x}_a$ and $\mathbf{x}_b$ belong to $\mathbf{x}_{v_{ab}}$, and there is a vertex v_{abc}, which is the first vertex $v \in \tau$ such that $\mathbf{x}_{v_{abc}}$ contains $\mathbf{x}_a$, $\mathbf{x}_b$, and $\mathbf{x}_c$. One proceeds as in the previous case, by extracting from the propagators in $\tilde{T}_{v_{ab}}$ and in $\tilde{T}_{v_{abc}}$ a term $\gamma^{h_r - h_{v_{abc}}} \gamma^{h_r - h_{v_{ab}}}$, which is sufficient to get again the bound (3.24); we omit the details. Hence we get the bound (3.19), by extracting the terms of the first order in $\bar{\lambda}_L$; an explicit calculation shows the constant $c_{L,\bar{m}}$ is bounded away from zero, uniformly in L.

Let us now analyze the function $G_\omega^{2,L}(\mathbf{x}-\mathbf{y})$, appearing in the l.h.s. of (3.18). By using (2.43) we find

$$G_\omega^{2,L}(\mathbf{x}-\mathbf{y}) = \sum_{h=h_L}^{0} \frac{1}{Z_h} \tilde{g}_\omega^{(h)}(\mathbf{x}-\mathbf{y}) + S_\omega^L(\mathbf{x}-\mathbf{y}) + G_\omega^{2,L,(uv)}(\mathbf{x}-\mathbf{y}), \tag{3.25}$$

where $\hat{\tilde{g}}_\omega^{(h)}(\mathbf{k}) = \hat{g}_\omega^{(h)}(\mathbf{k}) [\hat{Q}_\omega^{(h)}(\mathbf{k})]^2$, see (2.31), $G_\omega^{2,L,(uv)}(\mathbf{x}-\mathbf{y})$ is the contribution of the ultraviolet scales, which gives a negligible contribution for $|\mathbf{x}-\mathbf{y}| \to \infty$, and

$$S_\omega^L(\mathbf{x}-\mathbf{y}) = \sum_{n=1}^{\infty} \sum_{j_0=h_L-1}^{-1} \sum_{\substack{\tau \in \mathscr{T}_{j_0,n,2,0} \\ |P_{v_0}|=2}} S_{\tau,\omega}(\mathbf{x}-\mathbf{y}) \tag{3.26}$$

where $S_{\tau,\omega}(\mathbf{x}-\mathbf{y}) \equiv S_{2,0,\tau,(\omega,\omega)}(\mathbf{x},\mathbf{y})$.

We shall proceed as in the proof of (3.19). Let $\mathcal{T}_n$ the set of trees with n endpoints contributing to the r.h.s. of (3.26). Given $\tau \in \mathcal{T}_n$, there is a vertex $\bar{v}$, which is the higher one, such that $\mathbf{x}$ and $\mathbf{y}$ both belong to $\mathbf{x}_{\bar{v}}$, and we can extract from the propagators in $\tilde{T}_{\bar{v}}$ a factor $nC_N(1+\gamma^{h_{\bar{v}}}|\mathbf{x}-\mathbf{y}|)^{-N}$. Hence, we can bound $|S_{\tau,\omega}|$ with an expression which differs from the r.h.s. of (2.43) because of this factor and because we have to do an integration less, which gives a factor $\gamma^{2h_{\bar{v}}}$. It follows that, if $\mathbf{r}=\mathbf{x}-\mathbf{y}=(0,2r)$, with $r=\gamma^{-h_r}$, and we choose $N=2$,

$$|S_{\tau,\omega}(\mathbf{x}-\mathbf{y})| \leqslant C\gamma^{j_0}\gamma^{\bar{m}}(C\bar{\varepsilon}_L)^n\,\gamma^{2h_{\bar{v}}}\gamma^{-2(h_{\bar{v}}-h_L)}\prod_{i=1}^{2}\frac{\gamma^{-h_i}}{(Z_{h_i})^{1/2}}\prod_{\text{v not e.p}}\gamma^{-d_v}. \tag{3.27}$$

Note that, by (2.44), d_v can be equal to 0 for all vertices belonging to the path $\mathcal{C}$ connecting $\bar{v}$ with the root. However,

$$\gamma^{-(h_{\bar{v}}-j_0)}\prod_{\text{v not e.p}}\gamma^{-d_v}=\prod_{\text{v not e.p}}\gamma^{-\tilde{d}_v}, \tag{3.28}$$

with $\tilde{d}_v>0$ for all $v\in\tau$. Hence, if we use the analogous of bound (3.23) for summing over the scales of the two endpoints of type ϕ associated with the points $\mathbf{x}$ and $\mathbf{y}$, we get the bound

$$\begin{aligned}\sum_{\tau\in\mathcal{T}_n}|S_{\tau,\omega}(\mathbf{x}-\mathbf{y})| &\leqslant \sum_{\bar{h}=h_L}^{0}\ \sum_{\tau\in\mathcal{T}_n:\,h_{\bar{v}}=\bar{h}}|S_{\tau,\omega}(\mathbf{x}-\mathbf{y})|\\ &\leqslant (C\bar{\varepsilon}_L)^n\sum_{\bar{h}=h_L}^{0}\frac{\gamma^{\bar{h}}\gamma^{-2(\bar{h}-h_L)}}{Z_{h_{\bar{v}}}}\leqslant (C\bar{\varepsilon}_L)^n\frac{\gamma^{h_r}}{Z_{h_r}}.\end{aligned} \tag{3.29}$$

This bound, (3.25) and an explicit calculation of the first term in the r.h.s of (3.25) imply that, if $h_r=h_L+\bar{m}$,

$$\left|G^{2,L}_{\omega}(\omega\mathbf{r})-\omega\frac{\gamma^{h_r}}{Z_{h_r}}b_{L,\bar{m}}\right|\leqslant C\bar{\varepsilon}_L\frac{\gamma^{h_r}}{Z_{h_r}} \tag{3.30}$$

where $b_{L,\bar{m}}$ is a suitable constant, bounded away from 0, uniformly in L, for any fixed $\bar{m}$.

The bound (3.18) is a simple consequence of the bounds (3.19) and (3.30). ∎

We want now to show, by using Lemma 3.3, that the running coupling constants of the infinite volume Luttinger model are well defined and of order λ, up to $h=-\infty$. Let us define $\vec{v}_j=\vec{v}_j^{\infty}$, $\bar{\varepsilon}_j=\max_{j\leqslant i\leqslant 0}|\vec{v}_i|$, $\bar{\lambda}_j=\max_{j\leqslant i\leqslant 0}|\lambda_i|$, $\bar{\delta}_j=\max_{j\leqslant i\leqslant 0}|\delta_i|$. We shall prove the following Lemma.

Lemma 3.4. There are constants ε_1, c_2, and c_3 such that

$$|\vec{v}_1| \leqslant \varepsilon_1 \Rightarrow \bar{\lambda}_j \leqslant c_2 \varepsilon_1, \qquad \bar{\delta}_j \leqslant c_3 \varepsilon_1, \quad \forall j \leqslant 0. \tag{3.31}$$

Proof. In order to prove (3.31), we shall proceed by contradiction. To begin with, we suppose that there exists a $j \leqslant 0$ such that

$$\bar{\lambda}_{j+1} \leqslant c_2 \varepsilon_1 < |\lambda_j| \leqslant 2c_2 \varepsilon_1 \leqslant \varepsilon_0, \qquad |\bar{\delta}_j| \leqslant c_3 \varepsilon_1 \leqslant \varepsilon_0, \tag{3.32}$$

ε_0 being defined as in Lemma 3.2, and we prove that this is not possible, if ε_1, c_2 and c_3 are suitably chosen.

Let us consider the Luttinger model at finite volume L, such that $h_L = j - \bar{m}$ and $\bar{m}$ is a fixed integer, independent of L, such that $\gamma^{-j} \leqslant L/4$. The arguments used in the proof of Lemma 4.5 of ref. 5 and a rough bound on the difference between the infinite and finite volume propagators allow us to prove that there is $\tilde{\varepsilon} \leqslant \varepsilon_0$ such that, if $\tilde{\varepsilon}_L \equiv \max_{h_L \leqslant i \leqslant 0} \max\{|\vec{v}_i|, |\vec{v}_i^L|\} \leqslant \tilde{\varepsilon}$, then

$$|\vec{v}_i^L - \vec{v}_i| \leqslant C \tilde{\lambda}_{i+1} \tilde{\varepsilon}_{i+1}^{1/2} \frac{\gamma^{-i}}{L}, \qquad h_L \leqslant i \leqslant 0, \tag{3.33}$$

where $\tilde{\lambda}_s \equiv \max_{s \leqslant i \leqslant 0} \max\{|\lambda_i|, |\lambda_i^L|\}$, $\tilde{\varepsilon}_s \equiv \max_{s \leqslant i \leqslant 0} \max\{|\vec{v}_i|, |\vec{v}_i^L|\}$. We omit the details, which can be found in ref. 7, and are indeed very simple, once Lemma 4.5 of ref. 5 is understood.

Then, since $|\vec{v}_{i-1}^L - \vec{v}_i^L|$ and $|\vec{v}_{i-1} - \vec{v}_i|$ are of order $\tilde{\varepsilon}_i^2$ and $\bar{m}$ is a fixed number, it is very easy to prove, by an iterative argument, that the $\vec{v}_i^L$ for $h_L \leqslant i \leqslant 0$ and the $\vec{v}_i$ for $h_L \leqslant i < j$ are well defined, if the conditions (3.32) are satisfied, with any fixed values of c_2 and c_3 and ε_1 small enough; moreover

$$|\lambda_i^L - \lambda_i| \leqslant \frac{c_2}{2} \varepsilon_1, \qquad |\vec{v}_i^L| \leqslant (3c_2 + 2c_3)\, \varepsilon_1, \qquad h_L \leqslant i \leqslant 0. \tag{3.34}$$

On the other hand, by using (3.18), with $r = \gamma^{-j} = \gamma^{-h_L + \bar{m}}$, and (3.12), we get

$$|\lambda_j^L| \leqslant a_{\bar{m}}^{-1} \,[c_0 (\max_{h_L \leqslant i \leqslant 0} |\lambda_i^L|)(\max_{h_L \leqslant i \leqslant 0} |\vec{v}_i^L|) + c_1\, |\lambda|], \tag{3.35}$$

which implies, together with (3.34), that

$$|\lambda_j| \leqslant \frac{c_2}{2} \varepsilon_1 + a_{\bar{m}}^{-1} [c_0 (3c_2 + 2c_3)^2\, \varepsilon_1^2 + c_1 \varepsilon_1] \leqslant c_2 \varepsilon_1, \tag{3.36}$$

in contradiction with (3.32), if, for example, $c_2 = 4c_1/a_{\bar{m}}$ and $c_0(3c_2 + 2c_3)^2\, \varepsilon_1 / c_1 \leqslant 1$.

Note that the previous argument is valid also if we substitute in (3.32) the condition $\bar{\delta}_j \leqslant c_3 \varepsilon_1$ with $\bar{\delta}_{j+1} \leqslant c_3 \varepsilon_1 < \bar{\delta}_j \leqslant 2c_3 \varepsilon_1$. It follows that, in order to complete the proof of (3.31), we only have to prove that there is a contradiction in the hypothesis

$$\bar{\lambda}_h \leqslant c_2 \varepsilon_1 \leqslant \varepsilon_0, \qquad \bar{\delta}_{h+1} \leqslant c_3 \varepsilon_1 < \bar{\delta}_h \leqslant 2c_3 \varepsilon_1 \leqslant \varepsilon_0. \tag{3.37}$$

This result will be achieved by comparing the $\vec{v}_i$ of the Luttinger model, for $h \leqslant i \leqslant 0$, with the running coupling constants $\vec{v}_i^{(h)}$ the model with free measure (2.4) and interaction

$$\lambda_0 \int d\mathbf{x}\, \psi^{[h,0]+}_{\mathbf{x},+} \psi^{[h,0]-}_{\mathbf{x},+} \psi^{[h,0]+}_{\mathbf{x},-} \psi^{[h,0]-}_{\mathbf{x},-} + \delta_0 \sum_{\omega=\pm 1} i\omega \int dx\, \psi^{[h,0]+}_{\omega,x}\, \partial_x \psi^{[h,0]-}_{\omega,x}. \tag{3.38}$$

In fact, the analysis of ref. 9 implies that $|\vec{v}_0| \leqslant 2\,|\vec{v}_1| \leqslant 2\varepsilon_1$, if ε_1 is small enough, while the fact that the beta functions of the two models differ only because of the irrelevant terms on scale 0 implies that $\vec{v}_i^{(h)}$ is well defined for $h \leqslant i \leqslant 0$, if the conditions (3.37) are verified with ε_1 small enough, and $|\vec{v}_i - \vec{v}_i^{(h)}| \leqslant C\bar{\lambda}_h \bar{\varepsilon}_h^{1/2}$. Hence, it is easy to see that, to find a contradiction with (3.37), it is sufficient to prove that there is a constant c_3, such that

$$\bar{\lambda}_h^{(h)} \leqslant 2c_2 \varepsilon_1 \leqslant \varepsilon_0, \qquad \bar{\delta}_h^{(h)} \leqslant \varepsilon_0 \Rightarrow \bar{\delta}_h^{(h)} \leqslant c_3 \varepsilon_1 / 2. \tag{3.39}$$

In order to prove (3.39), we shall use the approximate gauge invariance of the model (3.38)–(2.4). In fact, by proceeding as in the proof of eq. (7.27) of ref. 5, which we refer to for definitions, it is possible to derive the following Ward identity, up to terms of the second order in the momenta:

$$\begin{aligned} 0 = {} & -\omega \delta_0 p - \hat{\Sigma}_{h,\omega}(\mathbf{k}-\mathbf{p}) + \hat{\Sigma}_{h,\omega}(\mathbf{k}) \\ & + [-ip_0 + \omega(1+\delta_0)\, p]\, \hat{\Gamma}_{h,\omega,\omega}(\mathbf{p},\mathbf{k}) + \hat{\Delta}_{h,\omega}(\mathbf{p},\mathbf{k}). \end{aligned} \tag{3.40}$$

Note that in ref. 5, the term proportional to δ_0 in (3.38) was included in the free measure, hence δ_0 does not appear in the Ward identity, as well as the term of the second order in the momenta, which we did not write explicitly in (3.40), since we shall use it only at zero momenta. In any case, the arguments sketched in ref. 5 and fully developed in ref. 6 allow to prove that

$$\left| \frac{\hat{\Delta}_{h,\omega}(\mathbf{p},\mathbf{k})}{(-ip_0 + \omega p)} \right| \leqslant c_4 \bar{\lambda}_h Z_h^{(h)}. \tag{3.41}$$

Let us define

$$\tilde{Z}\tilde{\delta}_h = \omega \frac{\partial \hat{\Sigma}_{h,\omega}}{\partial p}(0,0) - i\frac{\partial \hat{\Sigma}_{h,\omega}}{\partial p_0}(0,0), \qquad \tilde{Z}_h = 1 + i\frac{\partial \hat{\Sigma}_{h,\omega}}{\partial p_0}(0,0). \tag{3.42}$$

The analysis of ref. 5 implies that

$$\left|\frac{\tilde{Z}_h}{Z_h^{(h)}} - 1\right| \leqslant c_5 \bar{\lambda}_h, \qquad |\tilde{\delta}_h - \delta_h^{(h)}| \leqslant c_5 \bar{\lambda}_h^2. \tag{3.43}$$

On the other hand, if we put in (3.40) $p_0 = 0$ and take the limit $p \to 0$, we get

$$0 = -\omega\delta_0 + \frac{\partial \hat{\Sigma}_{h,\omega}}{\partial p}(\mathbf{k}) + \omega(1+\delta_0)\,\hat{\Gamma}_{h,\omega,\omega}(0,\mathbf{k}) + \lim_{p\to 0}\frac{\hat{\Delta}_{h,\omega}((p,0),\mathbf{k})}{p}, \tag{3.44}$$

while, if we put in (3.40) $p = 0$ and take the limit $p_0 \to 0$, we find

$$0 = i^{-1}\frac{\partial \hat{\Sigma}_{h,\omega}}{\partial p_0}(\mathbf{k}) - \hat{\Gamma}_{h,\omega,\omega}(0,\mathbf{k}) + \lim_{p_0\to 0}\frac{\hat{\Delta}_{h,\omega}((0,p_0),\mathbf{k})}{ip_0}. \tag{3.45}$$

It follows that

$$\tilde{Z}_h(\tilde{\delta}_h - \delta_0) = -\omega\lim_{p\to 0}\frac{\hat{\Delta}_{h,\omega}((p,0),0)}{p} + i(1+\delta_0)\lim_{p_0\to 0}\frac{\hat{\Delta}_{h,\omega}((0,p_0),)}{ip_0}, \tag{3.46}$$

implying, together with (3.41) and (3.43), that there exists a constant c_3, such that

$$|\delta_h^{(h)}| \leqslant \frac{c_3}{2}\varepsilon_1. \quad \blacksquare \tag{3.47}$$

Lemma 3.4 implies the vanishing of the Luttinger model Beta function in the form

$$|\vec{\beta}^l_{j,\lambda}(\vec{v}^L,\ldots,\vec{v}^L)| \leqslant C\,|\vec{v}^L|^2\,\gamma^{\theta j}, \qquad |\beta^l_{j,\delta}(\vec{v}^L,\ldots,\vec{v}^L)| \leqslant C\,|\vec{v}^L|^2\,\gamma^{\theta j}, \qquad 0<\theta<1, \tag{3.48}$$

which is what we need to complete the proof of Theorem (3.1). The proof is again by contradiction; assume that, for some $r \geqslant 2$

$$\beta^l_j(\vec{v}^L,\ldots,\vec{v}^L) = b_j(\vec{v}^L_j)^r + O((\vec{v}^L_j)^{r+1}), \tag{3.49}$$

with b_j a non vanishing constant. By Lemma 3.4 and Lemma 3.2 the running coupling constants $\vec{v}_j^L$ are analytic functions of $\vec{v}_1$

$$\vec{v}_j^L = \vec{v}_1 + \sum_{n=2}^{r} c_n^{(j)} (\vec{v}_j^L)^n + O((\vec{v}_j^L)^{r+1}) \tag{3.50}$$

and for any fixed j the sequence c_j^n is a bounded sequence. Inserting (3.50) in the analogous of (3.2) we find

$$\sum_{n=2}^{r} c_n^{(j-1)} (\vec{v}_j^L)^n = \sum_{n=2}^{r} c_n^{(j)} (\vec{v}_j^L)^n + \sum_{k=j+1}^{0} \sum_{n=3}^{r} d_{j,k}^n (\vec{v}_j^L)^n \tag{3.51}$$

where $\sum_{n=3}^{r} d_{j,k}^n (\vec{v}_j^L)^n$ represents the Taylor expansion of $D_{j,k}$ up to order r, and from (3.4)

$$|d_{j,k}^n| \leqslant \gamma^{-\eta(k-j)} C^n \sup_{2 \leqslant m \leqslant n-1} |c_m^{(j)} - c_m^{(k)}|. \tag{3.52}$$

Hence, inserting (3.52) in (3.51) we find

$$|c_n^{(j-1)} - c_n^{(j)}| \leqslant C^n \sum_{k=j+1}^{0} \gamma^{-\eta(k-j)} \sup_{2 \leqslant m \leqslant n-1} |c_m^{(j)} - c_m^{(k)}| \tag{3.53}$$

which, if $\lim_{j \to -\infty} c_n^{(j)} = c_n$, easily implies (by induction) that $|c_n^{(j-1)} - c_n| \leqslant C^n \gamma^{\eta/2j}$, for $2 \leqslant n \leqslant r-1$. This means that $|d_{j,k}^n| \leqslant \gamma^{\eta/4j} \bar{C}^n$ so that $c_n^{(j-1)} = c_n^{(j)} + b_r + O(\gamma^{\eta/4j})$ is necessarily a diverging sequence, and this is a contradiction.

4. WARD IDENTITIES AND DYSON EQUATION

4.1. Ward Identities

In the previous section we have proved the vanishing of the Beta function by using the exact solution of the Luttinger model. A very natural question is if it is possible to derive the same results directly in the framework of functional integration. As we said in the introduction, (3.1) is believed in the physical literature to follow from a set of Ward identities and Dyson equations. We derive rigorously such equations, but a consistent treatment of the cutoffs produces corrections to them which must be taken into account.

For technical reasons, which are explained in detail in ref. 6, one has to slightly modify the model described in Section 2.1 by substituting the function $[C_{h,0}(\mathbf{k})]^{-1}$ by a function $[C_{h,0}^{\varepsilon}(\mathbf{k})]^{-1}$, depending on a small parameter ε, which is equivalent as far the scaling properties of the theory are concerned, but is different from 0 for all allowed values of $\mathbf{k}$. This parameter has no essential role, since all bounds are uniform in it and one can take, at the end, the limit $\varepsilon \to 0$. Moreover, in this section we will only summarize our results, hence we will ignore this problem in the following.

By performing in (2.8) the *gauge transformation* $\psi^\sigma_{\mathbf{x},\bar\omega} \to e^{i\sigma\alpha_{\mathbf{x},\bar\omega}}\psi^\sigma_{\mathbf{x},\bar\omega}$ and $\psi^\sigma_{\mathbf{x},-\bar\omega} \to \psi^\sigma_{\mathbf{x},-\bar\omega}$ on the field with $\omega = \bar\omega$ (only), deriving with respect to $\alpha_{\mathbf{x},\bar\omega}$ and by putting $\alpha_{\mathbf{x},\bar\omega} = 0$, we get

$$0 = \frac{1}{Z(\phi, J)} \int P(d\psi)[-D_{\bar\omega}(\psi^+_{\mathbf{x},\bar\omega}\psi^-_{\mathbf{x},\bar\omega}) + \delta T_{\mathbf{x},\bar\omega} - \phi^+_{\mathbf{x},\bar\omega}\psi^-_{\mathbf{x},\bar\omega} + \psi^+_{\mathbf{x},\bar\omega}\phi^-_{\mathbf{x},\bar\omega}]$$

$$\cdot \exp\left\{ -V(\psi) + \sum_\omega \int d\mathbf{x}[J_{\mathbf{x},\omega}\psi^+_{\mathbf{x},\omega}\psi^-_{\mathbf{x},\omega} + \phi^+_{\mathbf{x},\omega}\psi^-_{\mathbf{x},\omega} + \psi^+_{\mathbf{x},\omega}\phi^-_{\mathbf{x},\omega}]\right\}, \quad (4.1)$$

where $Z(\phi, J) = \exp\{-\mathcal{W}(\phi, J)\}$,

$$D_\omega(\psi^+_{\mathbf{x},\omega}\psi^-_{\mathbf{x},\omega}) = \frac{1}{(L\beta)^2} \sum_{\mathbf{p},\mathbf{k}} D_\omega(\mathbf{p})\, e^{-i\mathbf{p}\mathbf{x}} \hat\psi^+_{\mathbf{k},\omega}\hat\psi^-_{\mathbf{k}+\mathbf{p},\omega}, \quad (4.2)$$

$D_\omega(\mathbf{p}) = -ip_0 + \omega p$ and

$$\delta T_{\mathbf{x},\omega} = \frac{1}{(L\beta)^2} \sum_{\mathbf{k}^+ \neq \mathbf{k}^-} e^{i(\mathbf{k}^+ - \mathbf{k}^-)\mathbf{x}} C_\varepsilon(\mathbf{k}^+, \mathbf{k}^-)\, \hat\psi^+_{\mathbf{k}^+,\omega}\hat\psi^-_{\mathbf{k}^-,\omega}. \quad (4.3)$$

$$C_\varepsilon(\mathbf{k}^+, \mathbf{k}^-) = [C^\varepsilon_{h,0}(\mathbf{k}^-) - 1]\, D_\omega(\mathbf{k}^-) - [C^\varepsilon_{h,0}(\mathbf{k}^+) - 1]\, D_\omega(\mathbf{k}^+). \quad (4.4)$$

In (4.2) (and always in the following) $\mathbf{p} = (p, p_0)$ is summed over momenta of the form $(2\pi n/L, 2\pi m/\beta)$, with n, m integers.

By deriving the r.h.s. of (4.1) with respect to $\phi^+_{\mathbf{y},\bar\omega}$ and $\phi^-_{\mathbf{z},\bar\omega}$ and then putting the external fields equal to 0, we obtain, in Fourier space, if $\mathbf{p} \neq 0$,

$$\hat G^{2,1}_\omega(\mathbf{p}, \mathbf{k}) = \frac{\hat G^2_\omega(\mathbf{k} - \mathbf{p}) - \hat G^2_\omega(\mathbf{k})}{D_\omega(\mathbf{p})} + \hat H^{2,1}_\omega(\mathbf{p}, \mathbf{k}), \quad (4.5)$$

where $\hat H^{2,1}_\omega(\mathbf{p}, \mathbf{k})$ is the Fourier transform of

$$H^{2,1}_\omega(\mathbf{x}; \mathbf{y}, \mathbf{z}) = \frac{\partial}{\partial J_{\mathbf{x},\omega}} \frac{\partial^2}{\partial\phi^+_{\mathbf{y},\omega}\,\partial\phi^-_{\mathbf{z},\omega}} \mathcal{W}_\Delta(\phi, J)|_{\phi = J = 0}, \quad (4.6)$$

with

$$W_\Delta(\phi, J) = \log \int P(d\psi)\, e^{-V(\psi) + \sum_\omega \int d\mathbf{x}[J_{\mathbf{x},\omega}T_{\mathbf{x},\omega} + \phi^+_{\mathbf{x},\omega}\psi^-_{\mathbf{x},\omega} + \psi^+_{\mathbf{x},\omega}\phi^-_{\mathbf{x},\omega}]}, \quad (4.7)$$

$$T_{\mathbf{x},\omega} = \frac{1}{(L\beta)^2} \sum_{\mathbf{k}^+ \neq \mathbf{k}^-} e^{i(\mathbf{k}^+ - \mathbf{k}^-)\mathbf{x}} \frac{C_\varepsilon(\mathbf{k}^+, \mathbf{k}^-)}{D_\omega(\mathbf{k}^+ - \mathbf{k}^-)} \hat\psi^+_{\mathbf{k}^+,\omega}\hat\psi^-_{\mathbf{k}^-,\omega}. \quad (4.8)$$

Analogously by performing fourth derivatives with respect to the ϕ fields

$$\hat{G}^{4,1}_{\omega}(\mathbf{p},\mathbf{k}_1,\mathbf{k}_2,\mathbf{k}_3)=D_{\omega}(\mathbf{p})^{-1}\,[\hat{G}^{4}_{\omega}(\mathbf{k}_1-\mathbf{p},\mathbf{k}_2,\mathbf{k}_3)-\hat{G}^{4}_{\omega}(\mathbf{k}_1,\mathbf{k}_2+\mathbf{p},\mathbf{k}_3)] +\hat{H}^{4,1}_{\omega}(\mathbf{p},\mathbf{k}_1,\mathbf{k}_2,\mathbf{k}_3), \tag{4.9}$$

where $H^{4,1}_{+}(\mathbf{p};\mathbf{k}_1,\mathbf{k}_2,\mathbf{k}_3)$ is the Fourier transform of

$$H^{4,1}_{\omega}(\mathbf{x};\mathbf{x}_1,\mathbf{x}_2,\mathbf{x}_3,\mathbf{x}_4)=\frac{\partial}{\partial J_{\mathbf{x},\omega}}\frac{\partial^2}{\partial\phi^{+}_{\mathbf{x}_1,\omega}\,\partial\phi^{-}_{\mathbf{x}_2,\omega}}\frac{\partial^2}{\partial\phi^{+}_{\mathbf{x}_3,-\omega}\,\partial\phi^{-}_{\mathbf{x}_4,-\omega}}\mathscr{W}_{\Delta}(\phi,J)|_{\phi=J=0}. \tag{4.10}$$

The Ward identities (4.5) and (4.9) *without the terms* $\hat{H}^{2,1}$ *and* $\hat{H}^{4.1}$ (which we can prove are non vanishing) are usually derived in the physical literature by various formal arguments, see for example refs. 8, 19, and 23. Indeed, if one removes the infrared cutoff (by putting $h=-\infty$) and neglects the correction terms $\hat{H}^{2,1}$ and $\hat{H}^{4,1}$, the identities (4.5), (4.9), (4.15), and (4.16) are the analogue, respectively, of Eqs. (3.9), (3.39), Figs. 8 and 9 of ref. 19, from which the vanishing of the density-density critical index and of the Beta function is claimed to follow.

4.2. Dyson Equation

It is possible to derive a Dyson equation which, combined with the second Ward identity, gives a relation between G^4, G^2, and $G^{2,1}$.

By (2.11), if $Z=\int P(d\psi)\exp\{-V(\psi)\}$ and $\langle\cdot\rangle$ denotes the expectation with respect to $Z^{-1}\int P(d\psi)\exp\{-V(\psi)\}$,

$$G^4_{+}(\mathbf{x}_1,\mathbf{x}_2,\mathbf{x}_3,\mathbf{x}_4)=\langle\psi^{-}_{\mathbf{x}_1,+}\psi^{+}_{\mathbf{x}_2,+}\psi^{-}_{\mathbf{x}_3,-}\psi^{+}_{\mathbf{x}_4,-}\rangle-G^2_{+}(\mathbf{x}_1,\mathbf{x}_2)\,G^2_{-}(\mathbf{x}_3,\mathbf{x}_4), \tag{4.11}$$

where we used the fact that $\langle\psi^{-}_{\mathbf{x},\omega}\psi^{+}_{\mathbf{y},-\omega}\rangle=0$.

Let $g_\omega(\mathbf{x})$ be the free propagator, whose Fourier transform is $g_\omega(\mathbf{k})=\chi_{h,0}(\mathbf{k})/(-ik_0+\omega k)$, see (2.4). Then, we can write the above equation as

$$\begin{aligned}
&G^4_{+}(\mathbf{x}_1,\mathbf{x}_2,\mathbf{x}_3,\mathbf{x}_4)\\
&\quad=-\lambda\int d\mathbf{z}\,g_{-}(\mathbf{z}-\mathbf{x}_4)\langle\psi^{-}_{\mathbf{x}_1,+}\psi^{+}_{\mathbf{x}_2,+}\psi^{-}_{\mathbf{x}_3,-}\psi^{+}_{\mathbf{z},-}\psi^{+}_{\mathbf{z},+}\psi^{-}_{\mathbf{z},+}\rangle\\
&\qquad+\lambda\,G^2_{+}(\mathbf{x}_1,\mathbf{x}_2)\int d\mathbf{z}\,g_{-}(\mathbf{z}-\mathbf{x}_4)\langle\psi^{-}_{\mathbf{x}_3,-}\psi^{+}_{\mathbf{z},-}\psi^{+}_{\mathbf{z},+}\psi^{-}_{\mathbf{z},+}\rangle\\
&\quad=-\lambda\int d\mathbf{z}\,g_{-1}(\mathbf{z}-\mathbf{x}_4)\langle[\psi^{-}_{\mathbf{x}_1,+}\psi^{+}_{\mathbf{x}_2,+}];[\psi^{-}_{\mathbf{x}_3,-}\psi^{+}_{\mathbf{z},-}\psi^{+}_{\mathbf{z},+}\psi^{-}_{\mathbf{z},+}]\rangle^{T}.
\end{aligned} \tag{4.12}$$

From (4.12) we get

$$
\begin{aligned}
-G^4_+(\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \mathbf{x}_4)
&= \lambda \int d\mathbf{z}\, g_-(\mathbf{z}-\mathbf{x}_4) \langle \psi^-_{\mathbf{x}_1,+}; \psi^+_{\mathbf{x}_2,+}; \rho_{\mathbf{z},+}\rangle^T \langle \psi^-_{\mathbf{x}_3,-}\psi^+_{\mathbf{z},-}\rangle \\
&\quad + \lambda \int d\mathbf{z}\, g_-(\mathbf{z}-\mathbf{x}_4) \langle \rho_{\mathbf{z},+}; \psi^-_{\mathbf{x}_1,+}; \psi^+_{\mathbf{x}_2,+}; \psi^-_{\mathbf{x}_3,-}; \psi^+_{\mathbf{z},-}\rangle^T \\
&\quad + \lambda \int d\mathbf{z}\, g_-(\mathbf{z}-\mathbf{x}_4) \langle \psi^-_{\mathbf{x}_1,+}; \psi^+_{\mathbf{x}_2,+}; \psi^-_{\mathbf{x}_3,-}; \psi^+_{\mathbf{z},-}\rangle^T \langle \rho_{\mathbf{z},+}\rangle,
\end{aligned}
\tag{4.13}
$$

where

$$
\rho_{\mathbf{x},\omega} = \psi^+_{\mathbf{x},\omega}\psi^-_{\mathbf{x},\omega}. \tag{4.14}
$$

The last addend is vanishing, since $\langle \rho_{\mathbf{z},\omega}\rangle = 0$ by the propagator parity properties. Then we get the identity, in terms of the Fourier transforms, as

$$
\begin{aligned}
-\hat{G}^4_+(\mathbf{k}_1, \mathbf{k}_2, \mathbf{k}_3)
&= \lambda \hat{g}_-(\mathbf{k}_4) \left[\hat{G}^2_-(\mathbf{k}_3)\, \hat{G}^{2,1}_+(\mathbf{k}_1-\mathbf{k}_2, \mathbf{k}_1) + \frac{1}{L\beta} \sum_{\mathbf{p}} G^{4,1}_+(\mathbf{p}; \mathbf{k}_1, \mathbf{k}_2, \mathbf{k}_3) \right],
\end{aligned}
\tag{4.15}
$$

where $\mathbf{k}_4 = \mathbf{k}_1 - \mathbf{k}_2 + \mathbf{k}_3$; see Fig. 2.

We shall call (4.15) the *Dyson equation* of our model.

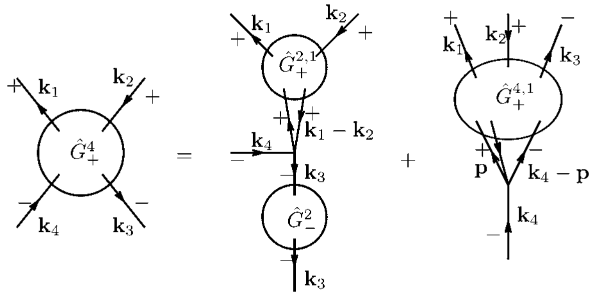

Fig. 2. Graphical representation of Dyson equation.

By using (4.9), (4.15) can be rewritten in the following way:

$$
\begin{aligned}
&-\hat{G}^4_+(\mathbf{k}_1, \mathbf{k}_2, \mathbf{k}_3) \\
&\quad = \lambda \hat{g}_-(\mathbf{k}_4) \left[\hat{G}^2_-(\mathbf{k}_3)\, \hat{G}^{2,1}_+(\mathbf{k}_1-\mathbf{k}_2, \mathbf{k}_2) + \frac{1}{L\beta} G^{4,1}_+(\mathbf{0}; \mathbf{k}_1, \mathbf{k}_2, \mathbf{k}_3) \right] \\
&\qquad + \lambda \hat{g}_-(\mathbf{k}_4) \frac{1}{L\beta} \sum_{\mathbf{p}\neq 0} \frac{\hat{G}^4_+(\mathbf{k}_1-\mathbf{p}, \mathbf{k}_2, \mathbf{k}_3) - \hat{G}^4_+(\mathbf{k}_1, \mathbf{k}_2+\mathbf{p}, \mathbf{k}_3)}{D_+(\mathbf{p})} \\
&\qquad + \lambda \hat{g}_-(\mathbf{k}_4) \frac{1}{L\beta} \sum_{\mathbf{p}\neq 0} \hat{H}^{4,1}_+(\mathbf{p}; \mathbf{k}_1, \mathbf{k}_2, \mathbf{k}_3).
\end{aligned} \tag{4.16}
$$

4.3. Consequences of Ward Identities and Dyson Equation

By using the tree expansion described in Section 2, it is possible to show that, in the model with cutoff function $C_{h,0}(\mathbf{k})^{-1}$, if $|\bar{\mathbf{k}}| = \gamma^h$ and $\bar{\varepsilon} = \max_{h \leqslant j \leqslant 0} |\vec{v}_j|$ is small enough,

$$\hat{G}^{2,1}_\omega(2\bar{\mathbf{k}}, \bar{\mathbf{k}}) = -\frac{Z^{(2)}_h}{Z^2_h D_\omega(\bar{\mathbf{k}})^2}\,[1+O(\bar{\varepsilon}^2)], \tag{4.17}$$

$$\hat{G}^2_\omega(\bar{\mathbf{k}}) = \frac{1}{Z_h D_\omega(\bar{\mathbf{k}})}\,[1+O(\bar{\varepsilon}^2)], \tag{4.18}$$

$$\hat{G}^4_+(\bar{\mathbf{k}}, -\bar{\mathbf{k}}, -\bar{\mathbf{k}}) = Z_h^{-2}\, |\bar{\mathbf{k}}|^{-4}\,[-\lambda_h + O(\bar{\varepsilon}^2)], \tag{4.19}$$

$$\left| \frac{1}{L\beta} \sum_{\mathbf{p}\neq 0} \hat{G}^{4,1}_\omega(\mathbf{p}, \bar{\mathbf{k}}, -\bar{\mathbf{k}}, -\bar{\mathbf{k}}) \right| \leqslant C \frac{|\bar{\varepsilon}|}{Z^2_h} \frac{Z^{(2)}_h}{Z_h} \gamma^{-3h} [\gamma^{C\bar{\varepsilon}|h|} - 1], \tag{4.20}$$

$$\left| \frac{1}{L\beta} \sum_{\mathbf{p}\neq 0} \frac{\hat{G}^4_+(\bar{\mathbf{k}}-\mathbf{p}, -\bar{\mathbf{k}}, -\bar{\mathbf{k}})|}{D_\omega(\mathbf{p})} \right| + \left| \frac{1}{L\beta} \sum_{\mathbf{p}\neq 0} \frac{\hat{G}^4_+(\bar{\mathbf{k}}, -\bar{\mathbf{k}}+\mathbf{p}, -\bar{\mathbf{k}})|}{D_\omega(\mathbf{p})} \right| \leqslant C\, |\bar{\varepsilon}| \frac{\gamma^{-3h}}{Z^2_h}, \tag{4.21}$$

$$\left| \frac{1}{L\beta} \sum_{\mathbf{p}\neq 0} \hat{H}^{4,1}_\omega(\mathbf{p}, \bar{\mathbf{k}}, -\bar{\mathbf{k}}, -\bar{\mathbf{k}}) \right| \leqslant C \frac{|\bar{\varepsilon}|}{Z^2_h} \frac{Z^{(2)}_h}{Z_h} \gamma^{-3h} [\gamma^{C\bar{\varepsilon}|h|} - 1]. \tag{4.22}$$

In Appendix A1 we give a short proof of (4.20), (4.21), (4.22); the proofs of (4.17), (4.18), (4.19) follow very easily from the bound (2.43) and are left to the reader.

Moreover, in refs. 6 the following bound was proved:

$$|\bar{\mathbf{k}}| = \gamma^h \Rightarrow C\gamma^{-2h}\bar{\varepsilon}^2 \frac{Z^{(2)}_h}{(Z_h)^2} \leqslant |\hat{H}^{2,1}_\omega(2\bar{\mathbf{k}}, -\bar{\mathbf{k}})| \leqslant 2C\gamma^{-2h}\bar{\varepsilon}^2 \frac{Z^{(2)}_h}{(Z_h)^2}. \tag{4.23}$$

Let us now discuss the main consequences of the previous bounds.

First of all (4.5), (4.17), (4.18), (4.23), together with Theorem (3.1), imply that $Z_h/Z_h^{(2)} = 1+O(\lambda^2)$. This is a very non trivial statement; in fact, $\lim_{h\to-\infty}\log[Z_{h-1}^{(2)}/Z_h^{(2)}] = \eta_2(\lambda)$ and $\lim_{h\to-\infty}\log[Z_{h-1}/Z_h] = \eta(\lambda)$, with $\eta_2(\lambda)$ and $\eta(\lambda)$ a priori different analytic functions of λ. However, the bound $Z_h/Z_h^{(2)} = 1+O(\lambda^2)$ implies that $\eta = \eta_2$. Note that this result is the same one would obtain if the correction term $\hat{H}^{2,1}$ were not present.

Let us see now if it possible to prove the vanishing of the Beta function (in the form of Theorem (3.1)) directly from the Dyson equation (4.15), as it is claimed in the physical literature, without any use of the Luttinger model exact solution. Indeed, *if one could neglect the term proportional to* $\hat{H}^{4,1}$, as it is usually done, (4.16), (4.21), (4.17), (4.18), and (4.19) would imply that $\lambda_h = \lambda Z_h^{(2)}/Z_h + O(\bar{\varepsilon}^2)$. Then, by a procedure similar to that used in Section 3, one could prove at the same time that the Beta function is vanishing and that $Z_h^{(2)}/Z_h = 1+O(\lambda^2)$. Note that this result could not be obtained, by using only the Dyson equation (4.15), because of the factor $[\gamma^{C\bar{\varepsilon}|h|}-1]$ present in the bound (4.20).

On the other hand, the presence of corrections spoils the above conclusion, because of the factor $[\gamma^{C\bar{\varepsilon}|h|}-1]$ *even* in the bound (4.22), contrary to our initial expectations. However, the bound (4.22) cannot be improved, unless cancellations at every order are taken into account, which is just what we want to avoid, as this would be equivalent to the original problem of proving cancellations directly in the Beta function. In fact, we can find terms in our expansion, which behave as $\bar{\varepsilon}^2 Z_h^{-2}\,|h|\,\gamma^{-3h}$, whose sum has the *right behavior* $\bar{\varepsilon}^2 Z_h^{-2}\gamma^{-3h}$.

The conclusion is that our procedure does not allow to prove rigorously the vanishing of the beta function through Ward identities and Dyson equation. The only hope is that one can prove that the correction terms go to 0 as the cutoff goes to infinity; however this is not a simple task, since it requires one is able to study the ultraviolet problem in the Thirring model.

APPENDIX A1

A1.1. Proof of (4.21)

The two sums on the l.h.s. of (4.21) can be studied in the same way. Let us consider, for example, $(L\beta)^{-1}\sum_{\mathbf{p}\neq 0}\hat{G}^4_+(\bar{\mathbf{k}}_1-\mathbf{p},\bar{\mathbf{k}}_2,\bar{\mathbf{k}}_3)/D_\omega(\mathbf{p})$, where $\mathbf{k}_i$ are momenta satisfying the relations

$$\bar{\mathbf{k}}_1 = \bar{\mathbf{k}}_4 = -\bar{\mathbf{k}}_2 = -\bar{\mathbf{k}}_3 = \bar{\mathbf{k}}, \qquad |\bar{\mathbf{k}}| = \gamma^h. \tag{A1.1}$$

Moreover, we shall consider only the case $L=\beta=\infty$, as the analysis of ref. 5 implies that the general case differs only by corrections which go to zero as $L, \beta \to \infty$. Hence, from now on, we shall substitute $(L\beta)^{-1} \sum_{\mathbf{p}\neq 0}$ with $(2\pi)^{-2} \int d\mathbf{p}$.

The support properties of the external propagator of momentum $\bar{\mathbf{k}}_1 - \mathbf{p}$ imply that $|\mathbf{p}| \leqslant \gamma + \gamma^h$, hence $|\mathbf{p}| \leqslant \gamma^2$, if γ^h is small enough, as we shall suppose (again only to simplify the notation). Hence we can write, defining $\chi_0(t)$ as in (2.5),

$$\begin{aligned}
&\int \frac{d\mathbf{p}}{(2\pi)^2} \frac{\hat{G}^4_+(\bar{\mathbf{k}}_1 - \mathbf{p}, \bar{\mathbf{k}}_2, \bar{\mathbf{k}}_3)}{D_\omega(\mathbf{p})} \\
&\quad = \int \frac{d\mathbf{p}}{(2\pi)^2} \chi_0(\gamma^{-2} |\mathbf{p}|) \frac{\hat{G}^4_+(\bar{\mathbf{k}}_1 - \mathbf{p}, \bar{\mathbf{k}}_2, \bar{\mathbf{k}}_3)}{D_\omega(\mathbf{p})} \\
&\quad = \int \frac{d\mathbf{p}}{(2\pi)^2} \chi_M(\mathbf{p}) \frac{\hat{G}^4_+(\bar{\mathbf{k}}_1 - \mathbf{p}, \bar{\mathbf{k}}_2, \bar{\mathbf{k}}_3)}{D_\omega(\mathbf{p})} + \sum_{h_p = h_{j_M}}^{2} \int \frac{d\mathbf{p}}{(2\pi)^2} f_{h_p}(\mathbf{p}) \frac{\hat{G}^4_+(\bar{\mathbf{k}}_1 - \mathbf{p}, \bar{\mathbf{k}}_2, \bar{\mathbf{k}}_3)}{D_\omega(\mathbf{p})},
\end{aligned} \tag{A1.2}$$

where j_M is defined so that $\gamma^{j_M} = M\gamma^h$, with $M = \gamma^2/(\gamma-1)$, and $\chi_M(\mathbf{p})$ is a smooth positive function with support in the ball $|\mathbf{p}| \leqslant \gamma^{j_M}$, such that $\chi_M(\mathbf{p}) + \chi_{[j_M, 0]}(\mathbf{p}) = 1$ for $|\mathbf{p}| \leqslant 1$. Note that j_M is defined so that, if $h_p \geqslant h_{j_M}$ and $f_{h_p}(\mathbf{p}) \neq 0$, then $|\bar{\mathbf{k}}_1 - \mathbf{p}| \in [\gamma^{h_p-2}, \gamma^{h_p+2}]$.

The bound (4.21) immediately follows from the following Lemma.

Lemma A1.1. If the momenta $\bar{\mathbf{k}}_i$ satisfy condition (A1.1), there exists a constant C such that

$$\int \frac{d\mathbf{p}}{(2\pi)^2} f_{h_p}(\mathbf{p}) \left| \frac{\hat{G}^4_+(\bar{\mathbf{k}}_1 - \mathbf{p}, \bar{\mathbf{k}}_2, \bar{\mathbf{k}}_3)}{D_\omega(\mathbf{p})} \right| \leqslant C\bar{\varepsilon} \frac{\gamma^{-3h}}{Z_h^2} \gamma^{-\frac{1}{2}(h_p - h)} \quad \text{if} \quad h_p \geqslant j_M, \tag{A1.3}$$

$$\int \frac{d\mathbf{p}}{(2\pi)^2} \chi_M(\mathbf{p}) \left| \frac{\hat{G}^4_+(\bar{\mathbf{k}}_1 - \mathbf{p}, \bar{\mathbf{k}}_2, \bar{\mathbf{k}}_3)}{D_\omega(\mathbf{p})} \right| \leqslant C\bar{\varepsilon} \frac{\gamma^{-3h}}{Z_h^2}. \tag{A1.4}$$

Proof. By (2.42), we can write

$$\hat{G}^{(4)}_+(\bar{\mathbf{k}}_1 - \mathbf{p}, \bar{\mathbf{k}}_2, \bar{\mathbf{k}}_3) = \sum_{n=1}^{\infty} \sum_{j_0 = h-1}^{-1} \sum_{\substack{\tau \in \mathcal{T}_{j_0, n, 4, 0} \\ |P_{v_0}| = 4}} \hat{G}_{4,\tau}(\bar{\mathbf{k}}_1 - \mathbf{p}, \bar{\mathbf{k}}_2, \bar{\mathbf{k}}_3), \tag{A1.5}$$

where $G_{4,\tau} \bar{S}_{4,0,\tau,\{+,+,-,-\}}$.

Look at the contribution $\hat{G}_{4,\tau}$, associated with a fixed tree τ, and suppose that $f_{h_p}(\mathbf{p}) \neq 0$, with $j_M \leqslant h_p$.The scale j_0 of v_0 has to be equal to h or $h+1$, since two of the external propagators, those with momenta $\bar{\mathbf{k}}_2$ and $\bar{\mathbf{k}}_3$, have scale h or $h+1$. Let us call h_1 and h_4 the scale indices of the other two external propagators, those with momenta $\bar{\mathbf{k}}_1 - \mathbf{p}$ and $\bar{\mathbf{k}}_4 - \mathbf{p}$; the definition of j_M implies that $|h_i - h_p| \leqslant 1$ for $i = 1, 4$. These two propagators are associated with two endpoints of type ϕ with scale $h_i + 1$ (see item 3 after Fig. 1); we shall call v_i the non trivial vertices of τ, of scale $\bar{h}_i \leqslant h_i$, immediately preceding them and v_p the higher vertex with $n_v^\phi = 2$. Of course the scale j_p of v_p has to be smaller that $\bar{h}_1$ and $\bar{h}_4$ and there is the possibility that $v_1 = v_4 = v_p$.

We shall consider three paths on the tree: the paths $\mathscr{C}_1$ and $\mathscr{C}_4$, connecting v_1 and v_4 with v_p, and the path $\mathscr{C}$ connecting v_p with v_0. By (2.43), if $v \in \mathscr{C}$ and $|P_v| = 4$, $d_v = 0$; in all the other cases $d_v > 0$. Hence, by using (2.43), the fact, proved in ref. 5, that

$$0 < 1 - Z_j / Z_{j-1} \leqslant C\bar{\varepsilon}^2, \tag{A1.6}$$

and the remarks that $|P_v| \geqslant 2$, $\forall v \in \tau$, and that on the path $\mathscr{C}$ there are at most $j_p - h + 2$ vertices, we get, for $\mathbf{p}$ in the support of $f_{h_p}(\mathbf{p})$,

$$\begin{aligned} &|\hat{G}_{4,\tau}(\bar{\mathbf{k}}_1 - \mathbf{p}, \bar{\mathbf{k}}_2, \bar{\mathbf{k}}_3)| \\ &\quad \leqslant (C\bar{\varepsilon})^n \frac{\gamma^{-2h}}{Z_h} \gamma^{\frac{1}{2}(j_p - h)} \prod_{v \notin \mathscr{C} \cup \mathscr{C}_1 \cup \mathscr{C}_4} \gamma^{-d_v} \\ &\quad\quad \cdot \frac{\gamma^{-h_1}}{\sqrt{Z_{h_1}}} \frac{\gamma^{-h_4}}{\sqrt{Z_{h_4}}} \left[\prod_{v \in \mathscr{C}_1 \cup \mathscr{C}_4} \left(\frac{\sqrt{Z_{h_v}}}{\sqrt{Z_{h_v - 1}}} \right) \gamma^{-d_v} \right] \left[\prod_{v \in \mathscr{C}} \left(\frac{Z_{h_v}}{Z_{h_v - 1}} \right) \gamma^{-1/2 - d_v} \right] \\ &\quad \leqslant (C\bar{\varepsilon})^n \gamma^{-2h_p} \frac{\gamma^{-2h}}{Z_h^2} \gamma^{\frac{1}{2}(j_p - h)} \sqrt{\frac{Z_{\bar{h}_1}}{Z_{h_p}}} \sqrt{\frac{Z_{\bar{h}_4}}{Z_{h_p}}} \prod_{v \notin \mathscr{C}} \gamma^{-d_v} \prod_{v \in \mathscr{C}} \gamma^{-1/2 - d_v}. \end{aligned} \tag{A.7}$$

If we fix the scales j_p, $\bar{h}_1$ and $\bar{h}_4$ of the vertices v_p, v_1 and v_4, we can sum over the sets P_v and the scale indices of the other vertices by using the general procedure explained in ref. 5, since there is a factor smaller than one in each vertex of the tree, as in the effective potential bounds. It is easy to see, by using (A1.6), that the sum over the remaining scale indices of $\gamma^{\frac{1}{2}(j_p - h)} \sqrt{Z_{\bar{h}_1} / Z_{h_p}} \sqrt{Z_{\bar{h}_4} / Z_{h_p}}$ (note that $j_p \leqslant \bar{h}_i \leqslant h_p + 1$ and $h \leqslant j_p \leqslant h_p + 1$) is bounded by $C\gamma^{(h_p - h)/2}$. Hence we get the bound (A1.3), since the integration over $\mathbf{p}$ gives a factor γ^{2h_p}, because of the support properties of $f_{h_p}(\mathbf{p})$, and $|D_\omega(\mathbf{p})|^{-1} \leqslant \gamma^{-h_p + 1}$ if $f_{h_p}(\mathbf{p}) \neq 0$.

The bound (A1.4) is obtained essentially in the same way. One has only to note that, if $\chi_M(\mathbf{p}) \neq 0$, $h_1 - h$ and $h_4 - h$ are smaller of a finite number only dependent on M, so that the path $\mathscr{C}$ can contain only a finite number of vertices. It follows that the sum over the trees with n normal endpoints of $|\hat{G}_{4,\tau}(\bar{\mathbf{k}}_1-\mathbf{p}, \bar{\mathbf{k}}_2, \bar{\mathbf{k}}_3)|$ can be bounded by $(C\bar{\varepsilon})^n \gamma^{-4h} Z_h^{-2}$, which implies (A1.4) since $\int \frac{d\mathbf{p}}{(2\pi)^2} \chi_M(\mathbf{p}) |D_\omega(\mathbf{p})|^{-1} \leqslant C\gamma^h$. ∎

A1.2. Proof of the Bound (4.20)

We have to study the quantities $\hat{G}_+^{4,1}(\mathbf{p}, \bar{\mathbf{k}}_1, \bar{\mathbf{k}}_2, \bar{\mathbf{k}}_3)$ and $(2\pi)^{-2}\int d\mathbf{p}\, \hat{G}_+^{4,1}(\mathbf{p}, \bar{\mathbf{k}}_1, \bar{\mathbf{k}}_2, \bar{\mathbf{k}}_3)$, with the momenta $\bar{\mathbf{k}}_i$ satisfying condition (A1.1).

The support properties of the external propagator of momentum $\bar{\mathbf{k}}_4-\mathbf{p}$ (see Fig. 2) imply that $|\mathbf{p}| \leqslant \gamma+\gamma^h$, hence we can proceed as in Section A1.1, by writing

$$\int \frac{d\mathbf{p}}{(2\pi)^2} \hat{G}_+^{4,1}(\mathbf{p}; \bar{\mathbf{k}}_1, \bar{\mathbf{k}}_2, \bar{\mathbf{k}}_3) = \int \frac{d\mathbf{p}}{(2\pi)^2} \chi_M(\mathbf{p})\, \hat{G}_+^{4,1}(\mathbf{p}; \bar{\mathbf{k}}_1, \bar{\mathbf{k}}_2, \bar{\mathbf{k}}_3) + \sum_{h_p=h_{j_M}}^{2} \int \frac{d\mathbf{p}}{(2\pi)^2} f_{h_p}(\mathbf{p})\, \hat{G}_+^{4,1}(\mathbf{p}; \bar{\mathbf{k}}_1, \bar{\mathbf{k}}_2, \bar{\mathbf{k}}_3). \tag{A.8}$$

The bound (4.20) immediately follows from the following Lemma.

Lemma A1.2. If the momenta $\bar{\mathbf{k}}_i$ satisfy condition (A1.1), there exists a constant C such that, if $f_{h_p}(\mathbf{p}) \neq 0$ and $h \wedge h_p = \min\{h, h_p\}$, then

$$|\hat{G}^{4,1}(\mathbf{p}, \bar{\mathbf{k}}_1, \bar{\mathbf{k}}_2, \bar{\mathbf{k}}_3)| \leqslant C\bar{\varepsilon}^2 \frac{\gamma^{-4h}}{Z_h^2} \gamma^{-h\wedge h_p} \frac{Z^{(2)}_{h\wedge h_p}}{Z_{h\wedge h_p}}; \tag{A1.9}$$

moreover,

$$\left|\int \frac{d\mathbf{p}}{(2\pi)^2} f_{h_p}(\mathbf{p})\, \hat{G}^{4,1}(\mathbf{p}, \bar{\mathbf{k}}_1, \bar{\mathbf{k}}_2, \bar{\mathbf{k}}_3)\right| \leqslant C\bar{\varepsilon}^2 \frac{\gamma^{-3h}}{Z_h^2} (1+C\bar{\varepsilon})^{h_p-h} \frac{Z_h^{(2)}}{Z_h}, \qquad \text{if} \quad h_p \geqslant j_M, \tag{A1.10}$$

$$\left|\int \frac{d\mathbf{p}}{(2\pi)^2} \chi_M(\mathbf{p})\, \hat{G}^{4,1}(\mathbf{p}, \bar{\mathbf{k}}_1, \bar{\mathbf{k}}_2, \bar{\mathbf{k}}_3)\right| \leqslant C\bar{\varepsilon}^2 \frac{\gamma^{-3h}}{Z_h^2} \frac{Z_h^{(2)}}{Z_h}. \tag{A1.11}$$

Proof. We can write

$$G^{4,1}(\mathbf{p}; \bar{\mathbf{k}}_1, \bar{\mathbf{k}}_2, \bar{\mathbf{k}}_3) = \sum_{n=1}^{\infty} \sum_{j_0=h-1}^{-1} \sum_{\substack{\tau \in \mathscr{T}_{j_0,n,4,1} \\ |P_{v_0}|=4}} \hat{G}_{4,1,\tau}(\mathbf{p}, \bar{\mathbf{k}}_1, \bar{\mathbf{k}}_2, \bar{\mathbf{k}}_3), \tag{A1.12}$$

where $G_{4,1,\tau}\bar{S}_{4,1,\tau,\{+,+,-,-,+\}}$. As in the previous sections, we analyze the consequences of the constraints on the scale indices, which follow from the momenta values. Let us consider first the case $f_{h_p} \neq 0$, with $h_p \geqslant j_M$.

First of all, we note that there are 3 endpoints of type ϕ and scale $\leqslant h+1$; hence $j_0 \leqslant h+1$. Let us call h_ϕ the scale of the first non trivial vertex, say v_ϕ, immediately preceding the fourth vertex of type ϕ, whose momentum is equal to $\bar{\mathbf{k}}_4 - \mathbf{p} = \bar{\mathbf{k}} - \mathbf{p}$. For simplicity, we shall suppose that this endpoint has scale $h_\phi + 1$ (see item 3 after Fig. 1); in fact the sum over the scale index of the propagator does not change the bound, as one can easily check. Moreover, we shall call h_J the maximum between the scales of the two propagators associated with the endpoint of type J, v_J the non trivial vertex (of scale h_J) immediately preceding it, and v_p the higher vertex with $n_v^\phi = n_v^J = 1$. Finally $\mathscr{C}_\phi$ and $\mathscr{C}_J$ will be the paths on the tree τ (possibly empty) which connect v_ϕ and v_J with v_p, while $\mathscr{C}$ will be the path connecting v_p with v_0. In Fig. 3 we plot a typical tree, by using empty circles to denote the special endpoints; the meaning of the dashed line is explained below.

The constraint on h_p implies that $|h_\phi - h_p| \leqslant 1$. On the other hand, since $\mathbf{p} = \mathbf{k}_1 - \mathbf{k}_2$, if $\mathbf{k}_1$ and $\mathbf{k}_2$ are the momenta of the two propagators emerging from the endpoint of type J, the definition of h_J and the support properties of the functions $f_j(\mathbf{k})$ imply that $2\gamma^{h_J+1} \geqslant \gamma^{h_p-1}$. It follows that $h_J \geqslant h_p - 2 - \log_\gamma 2$ and that the scale index j_p of the vertex v_p satisfies the condition $h \leqslant j_p \leqslant h_p + 1$. By proceeding as in the proof of (A1.7) and by using also the remark that $Z^{(2)}_{h_J}/Z_{h_J} \leqslant Z^{(2)}_{h_p}/Z_{h_p}\gamma^{C\bar{\varepsilon}^2(h_J-h_p)}$, we get

$$
\begin{aligned}
|\hat{G}_{4,1,\tau}(\mathbf{p}, \bar{\mathbf{k}}_1, \bar{\mathbf{k}}_2, \bar{\mathbf{k}}_3| &\leqslant (C\bar{\varepsilon})^n \gamma^{-h} \frac{\gamma^{-3h}}{(Z_h)^{3/2}} \prod_{v \notin \mathscr{C} \cup \mathscr{C}_\phi \cup \mathscr{C}_J} \gamma^{-d_v} \\
&\quad \cdot \frac{\gamma^{-h_\phi}}{\sqrt{Z_{h_\phi}}} \frac{Z^{(2)}_{h_J}}{Z_{h_J}} \left[\prod_{v \in \mathscr{C}_\phi \cup \mathscr{C}} \left(\frac{\sqrt{Z_{h_v}}}{\sqrt{Z_{h_v - 1}}} \right) \gamma^{-d_v} \right] \left[\prod_{v \in \mathscr{C}_J} \gamma^{-d_v} \right] \\
&\leqslant (C\bar{\varepsilon})^n \frac{\gamma^{-4h}}{Z_h^2} \gamma^{-h_p} \frac{Z^{(2)}_{h_p}}{Z_{h_p}} \prod_{v \notin \mathscr{C}_J} \gamma^{-d_v} \prod_{v \in \mathscr{C}_J} \gamma^{-d_v + C\bar{\varepsilon}^2}.
\end{aligned}
\tag{A1.13}
$$

If $v \in \mathscr{C}$ and $|P_v| = 2$, $d_v = 0$; in all the other cases $d_v > 0$. Moreover, if $\tilde{v}_p \leqslant v_p$ is the higher vertex belonging to $\mathscr{C}$, such that $|P_v| = 2$, all the vertices $v \in \mathscr{C}$ such that $v \leqslant \tilde{v}_p$ are trivial vertices with $|P_v| = 2$, except v_0 and, possibly, the vertex immediately following v_0 and belonging to $\mathscr{C}$; these vertices belong to a connected subpath $\tilde{\mathscr{C}}$ of $\mathscr{C}$, which can be empty and is represented as a dashed line in Fig. 3. This claim easily follows from the remark that, if $v \in \mathscr{C}$ and the cluster L_v has only two external ψ fields, one

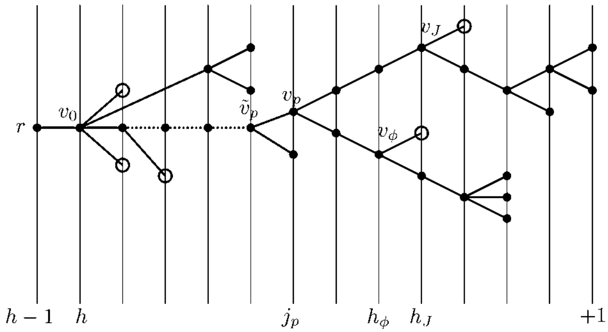

Fig. 3. A typical tree contributing to $G^{4,1}(\mathbf{p}; \bar{\mathbf{k}}_1, \bar{\mathbf{k}}_2, \bar{\mathbf{k}}_3)$.

of these fields is contracted in the external propagator of momentum $\bar{\mathbf{k}}_4 - \mathbf{p}$, while the other one, by momentum conservation, has to be contracted in a propagator with scale index equal to h or $h+1$, since the J field has momentum $\mathbf{p}$.

The previous considerations and the remark that the scale of the non trivial vertex v_ϕ is essentially fixed, imply that the sum over the non trivial vertices scale indices is equivalent to the sum over the *length* (difference between the scale indices of the extreme vertices) of all paths connecting two consecutive non trivial vertices, except $\tilde{\mathscr{C}}$. On the other hand, we can associate with each path of this type, of length m, a factor $\gamma^{-m/2}$, by extracting $\gamma^{-1/2}$ from the factor γ^{-d_v}, $d_v \geqslant 1$, associated with each vertex belonging to it in the bound (A1.13). The remaining factors $\gamma^{-d_v+1/2}$ are used to perform the sum over the sets P_v, as in ref. 5.

Finally an easy explicit calculation shows that the trees with $n=1$ cancel out exactly under the condition (A1.1) on the momenta. It follows that the sum over τ of the l.h.s. of (A1.13) can be bounded as

$$|\hat{G}^{4,1}(\mathbf{p}, \bar{\mathbf{k}}_1, \bar{\mathbf{k}}_2, \bar{\mathbf{k}}_3)| \leqslant C\bar{\varepsilon}^2 \frac{\gamma^{-4h}}{Z_h^2} \gamma^{-h_p} \frac{Z_{h_p}^{(2)}}{Z_{h_p}}. \tag{A1.14}$$

Let us now consider $(2\pi)^{-2} \int d\mathbf{p}\, f_{h_p}(\mathbf{p})\, \hat{G}_{4,1,\tau}(\mathbf{p}, \bar{\mathbf{k}}_1, \bar{\mathbf{k}}_2, \bar{\mathbf{k}}_3)$; its bound differs from the r.h.s. of (A1.13) for many reasons. First of all, there is a factor γ^{2h_p} related to the integration volume. Moreover, if $|\tilde{\mathscr{C}}| > 0$, $(2\pi)^{-2} \int d\mathbf{p}\, f_{h_p}(\mathbf{p})\, \hat{G}_{4,1,\tau}(\mathbf{p}, \bar{\mathbf{k}}_1, \bar{\mathbf{k}}_2, \bar{\mathbf{k}}_3)$ is of the form $G_1(\bar{\mathbf{k}}_4) \cdot \hat{g}^{(h)}(\bar{\mathbf{k}}_4) \cdot G_2(\bar{\mathbf{k}}_1, \bar{\mathbf{k}}_2, \bar{\mathbf{k}}_3, \bar{\mathbf{k}}_4)$, where $G_1(\bar{\mathbf{k}}_4)$ is a sum of graphs with an odd number of propagators of scale greater or equal to $\tilde{h}_p$, the scale of $\tilde{v}_p$. It follows that

$G_1(0)=0$, hence we can freely (i.e., without introducing new running coupling constants) regularize it in the usual way, so getting an improving factor $\gamma^{-(\tilde{h}_p-h)}$ in the bound.

The same parity argument allows us to show that, if $|P_v|=2,4$, we can perform all regularizations by increasing the Taylor expansion order by one unit, without introducing new running coupling constants; for the same reason, if $|P_v|+2n_v^J=6$, we can freely perform a first order regularization. Hence, we can improve the bound (2.43), by substituting d_v with

$$\bar{d}_v=\begin{cases} d_v+1 & \text{if} \quad n_v^\phi+n_v^J\leqslant 1, \quad |P_v|+2n_v^J\leqslant 6\\ d_v & \text{otherwise.}\end{cases} \tag{A1.15}$$

All these considerations, together with (A1.13), imply that

$$\begin{aligned} &\left|\int \frac{d\mathbf{p}}{(2\pi)^2} f_{h_p}(\mathbf{p})\, \hat{G}_{4,1,\tau}(\mathbf{p},\bar{\mathbf{k}}_1,\bar{\mathbf{k}}_2,\bar{\mathbf{k}}_3)\right|\\ &\quad\leqslant (C\bar{\varepsilon})^n \frac{\gamma^{-3h}}{Z_h^2}\gamma^{h_p-\tilde{h}_p}\frac{Z_{h_p}^{(2)}}{Z_{h_p}}\prod_{v\notin\mathscr{C}_J\cup\tilde{\mathscr{C}}}\gamma^{-\bar{d}_v}\prod_{v\in\mathscr{C}_J}\gamma^{-\bar{d}_v+C\bar{\varepsilon}^2}\\ &\quad\leqslant (C\bar{\varepsilon})^n \frac{\gamma^{-3h}}{Z_h^2}\frac{Z_{h_p}^{(2)}}{Z_{h_p}}\prod_{v\notin\mathscr{C}_J\cup\mathscr{C}}\gamma^{-d_v}\prod_{v\in\mathscr{C}_J}\gamma^{-d_v+C\bar{\varepsilon}^2}\prod_{v\in\mathscr{C}}\gamma^{-d_v^*}, \end{aligned} \tag{A1.16}$$

with $d_v^*=0$, if $|P_v|=2,4$, and $d_v^*>0$, if $|P_v|>4$.

Let us now perform the sum over the scale indices, by proceeding as in the proof of (A1.14). The bound is dominated by the trees such that $|P_v|=4$, if $\tilde{v}_p<v<v_p$, since these vertices can be non trivial. For these trees, the sum over the the scale indices associated with the non trivial vertices belonging to $\mathscr{C}\setminus\tilde{\mathscr{C}}$, each of them carrying at least a factor $C\bar{\varepsilon}$, can be bounded by $\sum_{r=0}^{h_p-\tilde{h}_p-1}\binom{h_p-\tilde{h}_p-1}{r}(C\bar{\varepsilon})^r$. Hence, it is not hard to deduce from (A1.16), by using also that $Z_{h_p}^{(2)}/Z_{h_p}\leqslant Z_h^{(2)}/Z_h\gamma^{C\bar{\varepsilon}^2(h_p-h)}$ and the remark before (A1.14) about the first order terms, the bound (A1.10).

Let us now suppose that $|\mathbf{p}|\leqslant M\gamma^h$. The previous analysis can be repeated, but the constraints on the scale indices are different. There is essentially no constraint on h_J, except the trivial one $h_J\geqslant h$, but $h_\phi-h\leqslant 1+\log(M+1)$, since $|\bar{\mathbf{k}}_4-\mathbf{p}|\leqslant(M+1)\,\gamma^h$, so that $h\leqslant h_p\leqslant h+\log_\gamma(M+1)$. Hence the length of the path $\mathscr{C}$ is bounded uniformly in τ and h, so that we get the bound (A1.11), as well as

$$|\hat{G}^{4,1}(\mathbf{p},\bar{\mathbf{k}}_1,\bar{\mathbf{k}}_2,\bar{\mathbf{k}}_3)|\leqslant C\bar{\varepsilon}^2\frac{\gamma^{-5h}}{Z_h^2}\frac{Z_h^{(2)}}{Z_h}. \tag{A1.17}$$

The bound (A1.9) immediately follows from (A1.17) and (A1.14). ∎

Sketch of the Proof for the Bound (4.22)

In ref. 6 we have shown that there is an expansion for $H_{\omega}^{4,1}$ similar to that used for $G_{\omega}^{4,1}$. The only important difference is that we have now three different special endpoints, associated with the field J and related to the field $T_{\mathbf{x},\omega}$ multiplying J in (4.7). These endpoints were called in ref. 6 of type J and *subtype* T, Z^+, and Z^-, respectively. Moreover, to the endpoints of type Z^+ and Z^- two new renormalization constants were associated, verifying the following bound

$$c_0\lambda^2 \leqslant \left|\frac{Z_j^{(3,+)}}{Z_j^{(2)}}\right| \leqslant 2c_0\lambda^2, \quad c_0\,|\lambda| \leqslant \left|\frac{Z_j^{(3,-)}}{Z_j^{(2)}}\right| \leqslant 2c_0\,|\lambda|, \quad j \in [h,-1]. \tag{A1.18}$$

Then, we can proceed as in the proof of the bound (4.20), by first proving a Lemma similar to Lemma A1.2. In fact, the trees with the special endpoint of subtype Z^+ can be treated exactly as the trees contributing to $\hat{G}_{\omega}^{4,1}$, as concerns the dependence on h, because the only difference is that the scale index of the special endpoint can not have the value $+1$. However, since $Z_h^{(3,+)}$ is of order $\bar{\varepsilon}^2$, these trees give a contribution of minimal order $\bar{\varepsilon}^4$ instead of $\bar{\varepsilon}^2$.

Let us now consider the trees with the special endpoint of subtype Z^-. The corresponding Feynman graphs are topologically equivalent to graphs contributing to $\hat{G}_{\omega}^{4,1}$, if we substitute the special endpoint with a generic graph with two external ψ fields of ω index opposite to that of the external field, see (153) of ref. 6. It easily follows, since $Z_h^{(3,-)}$ is of order $\bar{\varepsilon}$, that these trees are of minimal order $\bar{\varepsilon}^2$ and satisfy the same bound as the others, as concerns the dependence on h.

We still have to consider the trees with the special endpoint of subtype T, whose scale index h_T (see again ref. 6) is equal to $+1$, $h+2$ or $h+1$. If $h_T = +1$, by the usual arguments we can say that the value of the tree is exponentially depressed as $h \to -\infty$ with respect to the others; moreover, it is easy to see that it is of minimal order $\bar{\varepsilon}^2$. If $h_T \neq 0$, the value of the tree is depressed only by a factor $Z_h^{(2)}$, but it is still of minimal order $\bar{\varepsilon}^2$, since there is a cancellation between the first order contribution, as in the case of $\hat{G}_{\omega}^{4,1}$.

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